THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The two hundred nineteenth regular meeting of the Society was held at Columbia University on Saturday, October 29, 1921, extending through the usual morning and afternoon sessions. The attendance included the following forty members:

Archibald, Babb, Blake, Bowden, Cole, G. M. Conwell, Eisenhart, Fite, Philip Franklin, Gill, Glenn, Gronwall, Grove, Haskins, Hebbert, Joffe, Kasner, Kellogg, Kircher, Kline, Lamson, McDonnell, MacDuffee, MacNeish, H. H. Mitchell, Northcott, Oglesby, Pfeiffer, Post, Reddick, R. G. D. Richardson, Ritt, Seely, Siceloff, P. F. Smith, Veblen, H. S. White, Whittemore, E. C. Williams, J. W. Young.

At the meeting of the Council, the following thirty persons were elected to membership in the Society:

Mr. Clarence Raymond Adams, Harvard University;
Professor Edward Tankard Browne, Trinity College;
Professor Frank William Bubb, Washington University;
Mr. Curtis Corsair Carter, Bluffs, Scott County, Ill.;
Mr. Joel David Eshleman, University of Pennsylvania;
Mr. Matthew Moses Feldstein-Tartakovsky, Quartermaster Corps, U. S. Army;
Mr. Harold Joseph Gay, Worcester Polytechnic Institute;
Professor Lachlan Gilchrist, University of Toronto;
Professor Edward Sanford Hammond, Bowdoin College;
Mr. Edward H. Hezlett, Connecticut General Life Insurance Company;
Professor Joseph Ellis Hodgson, West Virginia University;
Miss Nelle Louise Ingels, Bureau of Statistics, Interstate Commerce Commission;
Mr. John Melvin Laird, Connecticut General Life Insurance Company;
Mr. Rudolph Ernest Langer, Harvard University;
Professor Michael Alexander Mackenzie, University of Toronto;
Professor Edmund Sewall Manson, Ohio State University;
Mr. Louis Edward Mensenkamp, High School, Freeport, Ill.;
Dr. Raymond Kurtz Morley, Worcester Polytechnic Institute;
Miss Eleanor Pairman, Broomieknowe, Scotland;
Miss Echo Dolores Pepper, University of Washington;
Mr. Charles Scott Porter, Worcester Polytechnic Institute;
Professor William Walter Rankin, Agnes Scott College;

Mr. Noah Bryan Rosenberger, Philadelphia, Pa.;
Mr. Norris Edward Sheppard, University of Toronto;
Professor George Eulas Foster Sherwood, University of California, Southern Branch;
Miss May Julia Sperry, Knox College;
Professor John Lighton Synge, University of Toronto;
Professor John Sidney Turner, Alabama Polytechnic Institute;
Mr. Correa Moylan Walsh, Bellport, L. I.;
Professor Thomas Orr Walton, Kalamazoo College.

One application for membership was received.

A committee consisting of Mr. S. A. Joffe and Professor P. H. Linehan was appointed to audit the accounts of the Treasurer for the current year. A list of nominations of officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election. It was voted to publish a List of Officers and Members for 1921–1922.

It was voted to accept the invitation of Harvard University to hold the annual meeting of 1922 at Cambridge.

Vice-president Cole presided at the morning session of the Society, relieved by Professor P. F. Smith in the afternoon. Titles and abstracts of the papers read at this meeting follow below. The papers of Mr. Rice, Dr. Walsh, Mr. Robinson, and Dr. Campbell were read by title. Professor R. L. Moore's papers were read by Professor Kline.

1. Professor J. K. Whittemore: *Total geodesic curvature.*

The author shows that the rate of turning, per unit length of arc of a curve on a surface, of the plane determined by the tangent to the curve and the normal to the surface (called by Professor Osgood* the bending of the curve) is given by

$$\sqrt{1/\rho_g^2 + 1/\tau_g^2},$$

where ρ_g and τ_g are the radii of geodesic curvature and geodesic torsion respectively.

2. Professor J. F. Ritt: *On the composition of polynomials.*

A polynomial $F(z)$ of degree greater than unity is here called *composite* if there exist two polynomials $\varphi_1(z)$ and $\varphi_2(z)$, each of degree greater than unity, such that $F(z) = \varphi_1[\varphi_2(z)]$.

* This BULLETIN, vol. 27, p. 391.

Otherwise $F(z)$ is called *prime*. Consider a decomposition of $F(z)$ into prime polynomials,

$$F = \varphi_1\varphi_2 \cdots \varphi_r,$$

in which each $\varphi_i(z)$ is understood to be substituted for z in the polynomial which precedes it. Two such decompositions are considered equivalent if the second can be obtained from the first by inserting suitable linear polynomials between the prime polynomials. In this paper the circumstances are completely determined under which a polynomial can have two distinct decompositions into prime polynomials. If there are two such distinct decompositions, they contain the same number of polynomials, and the degrees of the polynomials in one decomposition are the same as those in the other, except for the order in which they occur. The full statement of results is simple, but somewhat too long for an abstract.

3. Professor J. F. Ritt: *Complete determination of polynomials whose inverses can be expressed in terms of radicals.*

In an earlier paper the writer determined the structure of those Riemann surfaces with a prime number of sheets which support functions expressible in terms of radicals. For equations of composite degree, this paper determines all polynomials whose inverses can be expressed in terms of radicals. If a polynomial $F(z)$, with inverse so expressible, is decomposed into prime polynomials, in the manner explained in the preceding abstract, then each prime polynomial is either of the fourth degree, or else is of the form $\lambda_1\{\pi[\lambda_2(z)]\}$, where $\lambda_1(z)$ and $\lambda_2(z)$ are linear, and where $\pi(z)$ is either a prime power of z, or else a trigonometric polynomial of prime degree.*

4. Professor R. L. Moore: *Concerning continuous curves in the plane.*

In this paper the following theorems are established: (I) If N is a closed proper subset of a continuous curve M and two points of $M - N$ can be joined by a connected subset of $M - N$ then they can be joined by a simple continuous arc which is a subset of $M - N$; (II) If, in a plane, two points are separated from each other by a continuous curve M then they are separated from each other by a simple closed curve which is a subset of M. The first of these theorems is an extension of a theorem proved in a former paper (this BULLETIN, vol. 23 (1917), pp. 233–236).

* $\pi(z)$ is a trigonometric polynomial of degree n if $\cos nu = \pi(\cos u)$.

5. Professor R. L. Moore: *Concerning the relation of a continuous curve to its complement in space of three dimensions.*

According to a theorem of Schoenflies, in order that a point set M lying in a plane S should be a continuous curve it is necessary and sufficient that (1) every point of the boundary of any one of the domains complementary to M should be reachable "from all sides" with respect to that domain, (2) if $\epsilon > 0$ there do not exist in $S - M$ infinitely many mutually exclusive domains $R_1, R_2, R_3, \cdots$ all of diameter more than ϵ and such that each R_n has a portion of M as its boundary. In the present paper an example is given of a continuous curve in space of three dimensions which satisfies neither of these conditions. It is also shown that a similar criterion of Jordan and Schoenflies that a plane point set of more special type be a continuous curve is in the case of three dimensions neither necessary nor sufficient. It is true, however, that a closed, bounded and connected point set in a space of three dimensions is a continuous curve provided it is the common boundary of two mutually exclusive domains both of which are uniformly connected im kleinen. It is not sufficient to assume merely that one of these domains has this property.

6. Professor Edward Kasner: *An algebraic solution of Einstein's cosmological equations.*

Professor Kasner discusses the gravitational equations in Einstein's latest cosmological form $G_{ik} - \frac{1}{4}g_{ik}G = 0$. If we require the quaternary form ds^2 to be the sum of two binary forms, that is, the sum of the squared elements of two surfaces, then the only cosmological solution (neglecting the trivial euclidean form) is found to be

$$ds^2 = x_1^{-2}(dx_1^2 + dx_2^2) + x_3^{-2}(dx_3^2 + dx_4^2).$$

This represents a quartic manifold of four dimensions embedded in a 6-flat. The finite equations, in six cartesian coördinates, are

$$X_1^2 + X_2^2 + X_3^2 = 1, \qquad X_4^2 + X_5^2 + X_6^2 = 1.$$

This is probably the simplest solution of Einstein's equations which has thus far been found, and the first one (beyond the obvious flat and spherical spaces) which in its finite form is algebraic. In an earlier paper solutions have been found where the ten g_{ik} are algebraic functions, but these do not usually represent algebraic spreads. (See SCIENCE, Sept. 30, 1921, pp. 304–305, and a forthcoming paper in the MATHEMATISCHE ANNALEN.)

7. Dr. T. H. Gronwall: *On biharmonic functions.*

A biharmonic function satisfies the equation $\Delta\Delta u = 0$ or $\partial^4 u/\partial x^4 + 2\partial^4 u/\partial x^2 \partial y^2 + \partial^4 u/\partial y^4 = 0$ and is regular inside a curve C, the values of u and $\partial u/\partial n$ on C being given. Dr. Gronwall proves that when $\partial u/\partial n \geqq 0$ everywhere on C, then u has at least one maximum but no minimum inside C. A number of other qualitative results are also proved, most of which have immediate physical applications. Some of these results were conjectured, but not proved, by Hadamard in his prize memoir on elastic plates (MÉMOIRES DES SAVANTS ÉTRANGERS, 1909).

8. Mr. L. H. Rice: *General formulation of a combinatory method used by William Emerson and others.**

The method consists in so interpreting a given problem in combinations, in terms of element, entity, subclass, and subdivision, that the problem shall take the following form: Given, a stock containing k subclasses each included in the next, the ith subclass being composed of α_i elements. Required, the number C of combinations, each separable into k mutually exclusive subdivisions, the ith subdivision being composed of β_i elements taken from among the α_i elements (or composed of β_i entities which are in one to one correspondence with β_i of the α_i elements of the stock). The answer to the problem is then apparent:

$$C = \binom{\alpha_1}{\beta_1}\binom{\alpha_2-\beta_1}{\beta_2}\binom{\alpha_3-\beta_2-\beta_1}{\beta_3}\cdots\binom{\alpha_k-\beta_{k-1}-\cdots-\beta_1}{\beta_k}.$$

9. Dr. J. L. Walsh: *A theorem on loci connected with cross-ratios.*

The following theorem, formerly proved for real λ, is now extended to non-real λ: If the respective loci of the points z_1, z_2, z_3 are circular regions, then the locus of the point z_4 determined by the constant cross-ratio $\lambda = (z_1, z_2, z_3, z_4)$ is also a circular region.

* (1) William Emerson, *The Doctrine of Combinations, Permutations, and Composition of Quantities*, London, 1770. (2) L. Oettinger, ARCHIV FÜR MATHEMATIK, 15 (1850), p. 241, § 6. (3) L. H. Rice, *Coefficient of the general term in the expansion of a product of polynomials*, this BULLETIN, vol. 27 (1921), p. 344.

10. Mr. L. B. Robinson: *A generalization of the notion of covariants.*

Riquier has generalized the theory of complete systems. Mr. Robinson shows that a generalized complete system

$$\frac{\partial w_i}{\partial x_j} = \sum_{r=1}^{n} A_{rj} \frac{\partial w_i}{\partial t_r} + U_{ij}$$

can be utilized to generalize the notion of covariants. The generalized covariants are solutions of the above system and can be obtained by a finite number of differentiations or integrations.

11. Dr. G. A. Campbell: *Inductances of grounded circuits.*

Any network of conductors located on the surface of the earth with which it is conductively connected at any number of points will, for direct currents, have self and mutual inductances which are equal to the Neumann integral extended over the network alone. In other words, that portion of the complete Neumann integral for closed circuits which involves the return currents through the earth vanishes. The earth is assumed to be flat, of infinite extent, of unit permeability and of uniform conductivity.

R. G. D. RICHARDSON,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION

The thirty-eighth regular meeting of the San Francisco Section was held at the University of California on Saturday, October 22, 1921. Professor Lehmer presided at the earlier part of the meeting, later relieved by Professor Allardice. The total attendance was twenty-five, including the following seventeen members of the Society:

Alderton, Allardice, Bernstein, Blichfeldt, Buck, Cajori, Daus, Edwards, Haskell, Hoskins, Lehmer, Moreno, F. R. Morris, Noble, T. M. Putnam, Pauline Sperry, A. R. Williams.

The following officers were elected for the year: Chairman, Professor Allardice; Secretary, Professor Bernstein; programme committee, Professors Blichfeldt, Lehmer, Bernstein.

The dates of the next two meetings were fixed as April 8, 1922, and October 21, 1922.

Titles and abstracts of the papers read at this meeting follow.

Mr. Wong was introduced by Professor Lehmer. In the absence of the authors, the papers of Professor Moritz and Professor Bell were read by title.

1. Professor Thomas Buck: *A class of asymptotic solutions of the restricted problem of three bodies.* Preliminary report.

The solutions in question are of the type considered by Poincaré, and are asymptotic to the Lagrangian equilateral triangle solutions.

2. Professor Florian Cajori: *Spanish and Portuguese symbols for "thousands."*

The Spanish symbol named *calderón*, designating "thousand," found in Spanish and Spanish-American manuscripts as well as in printed books as far back as the fifteenth century, is identified by Professor Cajori as being the same as a symbol for "thousand" used in northwestern Italy during the fifteenth century. The calderón, the author points out, is probably a modified form of the Roman symbol CIↃ for "thousand." The Portuguese *cifrão* resembles our $, and is used like the Spanish calderón, having probably the same origin.

3. Professor Florian Cajori: *History of notations of the calculus.*

The history of early calculus notations is given in detail in our histories of mathematics and books relating to Newton and Leibniz. Cantor's *Geschichte der Mathematik*, vol. 4, for the years 1759–1799, pays little attention to calculus notations of that period, and no detailed history of the nineteenth century notations exists. The present paper endeavors to cover the period since about 1740.

4. Mr. B. C. Wong: *A theorem on double points.*

The following theorem is proved: The total number D of double points on a composite curve of order n made up of r curves is greater by $r - 1$ than the maximum number D of double points on a proper curve of the same order diminished by the sum P of the deficiencies of the r curves; that is $D = D' + r - 1 - P$.

5. Professor B. A. Bernstein: *On complete independence of Hurwitz's postulates for abelian groups and fields.*

Professor Bernstein discusses the question of complete independence of each of the postulate-sets for abelian groups and fields presented by Hurwitz in the ANNALS OF MATHEMATICS, (2), vol. 15 (1913), p. 93.

6. Dr. A. R. Williams: *On finding the rate of interest on a given investment.*

The writer shows that in finding the rate of interest yielded by a redeemable security it is generally advantageous to reduce the equation to the form $i = f(i)$, where $f(i)$ is a rational function of i, whose rate of change with respect to i can be calculated if desired. It is then easy to find by trial a value of i to make the two members agree to any desired number of places. The method is self-checking, and is particularly easy of application when $f(i)$ is a decreasing function, as will be the case when the redemption price is greater than the purchase price.

7. Professor R. E. Moritz: *Anti-ratientiation; definition and fundamental relations.*

In a paper published in the AMERICAN JOURNAL, vol. 24, pp. 257–302, Professor Moritz considered certain general limiting processes of the nth order (ratientiation) which include the differentiation and quotientiation processes as special cases corresponding to $n = 0$ and $n = 1$ respectively. In the present paper the corresponding inverse processes are defined and their fundamental relations developed. The results will be embodied in a more comprehensive paper which will appear in the TÔHOKU JOURNAL.

8. Professor E. T. Bell: *Extensions of Dirichlet multiplication and Dedekind inversion.*

In this paper, which appears in this number of the BULLETIN, the author develops the theory of fields in which multiplication and division are extensions of processes defined by Cauchy, Dirichlet and Dedekind for series. The theory has been devised for the specific purpose of simplifying and correlating much of the material summarized in volume I, chapters V, X, XIX of Dickson's *History of the Theory of Numbers.*

9. Professor E. T. Bell: *A qualitative theory of singly infinite series and products.*

The object of the theory in Professor Bell's second paper is to continue that of his first and to determine under what conditions, useful in the theory of numbers, infinite series and infinite products can be manipulated to yield arithmetic theorems without any reference to their convergence or divergence.

Three kinds of multiplication of series and their inverses are discussed in detail. The third, Euler multiplication and division, is new, and connects the Dirichlet and Cauchy multiplication of series with the expansion of products of an infinite number of infinite series. Complete algebras of these processes are constructed, and each is shown to be abstractly identical with common algebra.

10. Professor E. T. Bell: *Arithmetical equivalents for a remarkable relation between theta functions.*

In a former paper, to appear shortly in the GIORNALE DI MATEMATICHE, Professor Bell gave a complete system of arithmetic equivalents for the equation of three terms in elliptic functions.

The present paper gives equivalents of a remarkable relation, due to Hermite and Enneper, which is complementary to the equation of three terms.

11. Professor E. T. Bell: *The arithmetic of the nodes and tropes on a Kummer surface.*

In a previous paper, to appear in the AMERICAN JOURNAL, Professor Bell remarked that the configuration of nodes and tropes on the Kummer surface is formally equivalent, in the sense of mathematical logic, to a system of theorems on the simultaneous representation of a pair of integers each as a sum of four squares.

This paper presents the equivalence which, for lack of space, was omitted from the other.

B. A. BERNSTEIN,
Secretary of the Section.

THE SIMPLE GROUP OF ORDER 2520

BY G. A. MILLER

Extract from a letter to F. N. Cole

If a second simple group of order 7!/2 existed it could not be represented as a primitive group whose degree is less than twenty-one since all these primitive groups have been determined. The number of its subgroups of order 7 would be 120, for the only other divisor of 7!/2 which is of the form $1 + 7k$ and greater than 20 is 36. It is easy to prove that such a simple group could not involve exactly 36 subgroups of order 7, as follows.

If a simple group of order 7!/2 contained exactly 36 subgroups of order 7, it could be represented as a transitive group G on 36 letters representing the permutations of these 36 subgroups. Its subgroup G_1 composed of all its substitutions omitting one letter would be of order 70. It would therefore involve a cyclic subgroup of order 35 which would be regular, since the substitutions of order 7 would be regular. The subgroup G_1 could not be dihedral, since it could not involve negative substitutions. For the same reason the substitutions of order 2 could not transform the substitutions of order 5 into themselves and the substitutions of order 7 into their inverses. If these substitutions of order 2 could transform the substitutions of order 7 into themselves and the substitutions of order 5 into their inverses, G_1 would involve 5 conjugates of order 2. But this is impossible, since $36 \cdot 5$ is not divisible by 8.

Having proved that if the group in question existed it would contain 120 subgroups of order 7, we proceed to consider its subgroups of order 9. The number of these subgroups would be divisible by 35. In fact, if an operator of order 5 or an operator of order 7 could transform a subgroup of order 9 into itself it would be commutative with each of its operators. It was proved above that an operator of order 7 cannot be

commutative with every operator of a subgroup of order 9. If an operator of order 5 were commutative with every operator of such a subgroup, the number of the subgroups of order 5 would be 56. Hence the group in question could be represented as a transitive group G of degree 56, in which G_1 would be an abelian group of order 45 involving substitutions of degree 55 and order 5. The subgroup of order 9 contained in G_1 could not have a transitive constituent of degree 9, since $8 \cdot 56$ is not divisible by 11. For a similar reason it could not have a substitution of degree 30. Hence G could not involve exactly 56 subgroups of order 5.

Since the number of subgroups of order 9 would be divisible by 35, this number would be either 70 or 280. In the latter case, the group in question could be represented as a transitive group G of degree 280, in which G_1 would be of degree 279 and of order 9. If all the substitutions besides the identity of G_1 were of degree 279, G would be of class 279 and hence could not be simple. Hence G_1 would involve at least three transitive constituents of degree 3. If a substitution of G_1 were of degree 276, it would be invariant under 4 and only 4 subgroups of order 9, and hence under a subgroup of order 36 which would contain an invariant subgroup of order 4. Hence the group under consideration could be represented as a transitive group G of degree 70 in which the G_1 of order 36 contains four subgroups of order 9.

The invariant subgroup of order 4 contained in this G_1 would be invariant under a larger group since it would be invariant under a group of order 8. This larger group would contain more than 4 subgroups of order 9 and the number of these subgroups would be divisible by 8. As this is impossible, it has been proved that if G contains 280 subgroups of order 9 the G_1 of the transitive group according to which they are transformed cannot contain a substitution of degree 276.

If this G_1 contained a substitution of degree 273, there would be a subgroup of order 63 which would involve an invariant subgroup of order 7, but this is contrary to the result obtained above. If it contained a substitution of degree 270, there

would be a subgroup of order 90 which would have a quotient group of order 30. As the latter could not contain ten subgroups of order 3, this is impossible. Since no substitution of this G_1 could be of a lower degree than 270, it has been proved that the number of subgroups of order 9 contained in G is exactly 70, and that G can therefore be represented as a transitive group of degree 70 representing the transformations of its subgroups of order 9. The G_1 of this G is therefore of degree 69 and of order 36 and contains a single subgroup of order 9.

The substitutions in this subgroup of order 9 cannot all be of degree 69, since 9 does not divide 69. Hence this group of order 9 contains at least one substitution whose degree does not exceed 66. If it contains no substitution besides the identity of lower degree, it has two and only two transitive constituents of degree 3. In this case G_1 must have a transitive constituent of degree 6, since it cannot involve an invariant subgroup of order 3 and of degree less than 69. Moreover, the transitive constituent of degree 6 must be of order 36, as otherwise G_1 would involve an invariant substitution of order 2 whose degree could not exceed 36. The four subgroups of order 9 which are transformed into themselves by a substitution of order 3 and of degree 66 must therefore generate a group of order 36 in which this substitution is invariant, since a subgroup of order 18 in a transitive group of degree 6 and order 36 does not contain an invariant substitution of degree 3. It was proved above that this subgroup of order 36 contains an invariant subgroup of order 4 involving three conjugate substitutions of order 2.

As this subgroup of order 36 could have only 9 operators in common with G_1, its invariant subgroup of order 3 would be transformed into itself by 9 operators of the latter subgroup which are not found in the former subgroup of order 36. Hence we may confine our attention to the case when this invariant subgroup of order 3 would be transformed into itself by exactly 72 operators of G, and when G would involve a single set of 35 conjugate subgroups of order 3. Hence G

may be supposed to be represented as a primitive group of degree 35 and its G_1 has an invariant subgroup of order 3 whose substitutions are invariant under a G_{36} and are of degree 30. This G_{36} contains 26 substitutions of order 3 of which 10 are of degree 30 and 16 of degree 33, while its three substitutions of order 2 are of degree 28 since it has three transitive constituents, of degrees 18, 12, and 4, respectively. All of these conclusions follow from the properties of the subgroup of order 36 considered in the preceding paragraph. Each of the subgroups of order 9 has three regular constituents of order 9.

The G_1 of order 72 has also three transitive constituents, one of which is the symmetric group of degree 4. It involves 21 substitutions of order 2 and of degree 28 and 18 of order 4 and of degree 34. As all the non-invariant non-cyclic subgroups of order 4 contained in this G_1 are conjugate, this G_1 is completely determined. Its generalized dihedral subgroups of order 18 are therefore also determined. Hence the group of order 36 which contains such a dihedral group is also determined, and there is no such group besides the known simple G, which can be represented on 7 letters. It should be noted that there is a substitution of order 2 which is commutative with every substitution of the transitive constituent of degree 18 and involves only the letters of this constituent, but is not found therein. This permutes its two regular constituents of order 9 which appear in the dihedral group of order 18 noted above and it is on this account that the group of order 36 in question is completely determined. This group contains a constituent of degree 2 and together with G_1 generates G.

It has now been proved that if there were a second simple group of order $7!/2$, G_1 would involve substitutions whose degree would be less than 66. As the subgroup of order 9 in G_1 could not contain exactly five transitive constituents of degree 3 it would have to involve at least 8 such constituents. If G_1 contained a substitution of order 3 and of degree 63, the seven subgroups of order 9 which would be transformed into themselves by this substitution would generate a group in which this substitution is invariant, and hence this group would con-

tain exactly 7 subgroups of order 9. Its order would therefore be 63, 126, or 252. This is impossible as each of these groups would involve only one subgroup of order 7 since each of its subgroups of order 7 would be transformed into itself by at least 21 substitutions.

It remains only to consider the case when G_1 would contain a substitution of order 3 and of degree 60 without involving such a substitution of degree 63. The order of the group formed by all the substitutions of G which would be commutative with this substitution of order 3 would be 90. This group of order 90 would transform its ten subgroups of order 9 according to a transitive group of order 30 and of degree 10. Since this transitive group does not exist,* we have arrived at nothing but contradictions by assuming the existence of a second simple group of order $7!/2$ and hence such a group is actually proved to be non-existent.

THE UNIVERSITY OF ILLINOIS.

A THEOREM OF OSCILLATION

BY W. E. MILNE

In an investigation of the oscillations of aerial bombs a need was found for the following proposition. Both the theorem and its proof are modelled after a similar theorem and proof by Osgood.†

THEOREM. *Let $\varphi(t)$ be positive, continuous, monotonically increasing, and bounded in the interval $T \leqq t < \infty$, and let m and M be two positive constants such that $m < \varphi(t) < M$ for $t > T$. Let $f(y)$ be an odd, monotonically increasing function, satisfying the Lipschitz condition*

$$|f(y_1) - f(y_2)| < K|y_1 - y_2|, \qquad K > 0,$$

in an interval $-a \leqq y \leqq +a$, $a > 0$. Let y be a solution of the differential equation

$$\frac{d^2y}{dt^2} + \varphi(t)f(y) = 0 \tag{1}$$

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* Cf. F. N. Cole, QUARTERLY JOURNAL, vol. 27 (1895), p. 40.

† This BULLETIN, vol. 25 (1919), pp. 216–221.

subject to the conditions

$$(2)\quad \frac{dy}{dt} = 0,\ y = y_1,\ |y_1| < a,\ f(y_1) \neq 0,\ \textit{when}\ t = t_1 > T.$$

Then y oscillates an infinite number of times in the interval $t_1 < t < +\infty$ and the amplitudes decrease monotonically but do not approach zero.

Proof: Let us extend* the definition of $f(y)$ by the formulas

$$f(y) = f(a), \qquad \text{when} \qquad y > a,$$
$$f(y) = f(-a), \qquad \text{when} \qquad y < -a.$$

The function so extended satisfies the Lipschitz condition.

With the hypotheses thus extended, there exists† a unique function $y(t)$, continuous together with its first two derivatives, which satisfies (1) and (2) in the interval $t_1 \leqq t < \infty$. Now consider the case in which y_1 is positive. Then, at t_1, d^2y/dt^2 is negative and remains negative as long as y is positive. Since

$$v = \frac{dy}{dt} = \int_{t_1}^{t} \frac{d^2y}{dt^2}\,dt, \qquad t > t_1,$$

we see that v is negative as long as y is positive. Therefore the graph of $y(t)$ as a function of t is concave downward with negative slope to the right of t_1, and therefore must cut the axis at a finite point $t_1' > t_1$. Let v_1 be the corresponding value of v. Now multiply (1) by $2dy$ and integrate, obtaining

$$(3)\qquad v^2 = -2\int_{y_1}^{y'} \varphi(t) f(y)\,dy.$$

At t_1' this becomes

$$v_1^2 = 2\int_{0}^{y_1} \varphi(t) f(y)\,dy.$$

Since in the interval $t_1 \leqq t \leqq t_1'$ we have by hypothesis $\varphi(t_1) \leqq \varphi(t) \leqq \varphi(t_1')$, it follows that

$$v_1^2 \leqq 2\varphi(t_1')\int_0^{y_1} f(y)\,dy,$$

$$v_1^2 \geqq 2\varphi(t_1)\int_0^{y_1} f(y)\,dy.$$

Now let

$$\int_0^y f(y)\,dy = F(y).$$

* This extension is made for convenience in establishing the existence of the solution. Actually the definition of $f(y)$ outside the interval $-a \leqq y \leqq a$ is immaterial.

† Bliss, PRINCETON COLLOQUIUM, p. 93.

Then $F(y)$ is even and continuous for $|y| < a$, is monotonically increasing in the interval $0 < y < a$, and vanishes at the origin. With this notation the above inequalities become

$$\begin{cases} v_1^2 \leqq 2\varphi(t_1')F(y_1), \\ v_1^2 \geqq 2\varphi(t_1)F(y_1). \end{cases} \tag{4}$$

At t_1' v is negative; hence, immediately to the right of t_1' y is negative, and therefore d^2y/dt^2 is positive. Moreover as long as v is negative, d^2y/dt^2 is monotonically increasing, as equation (1) shows. Then since

$$v = v_1 + \int_{v_1}^{v} \frac{d^2y}{dt^2} dt,$$

it is clear that v must vanish for a finite value of t, $t = t_2 > t_1'$. Let the corresponding value of y be y_2. Now from (3)

$$v_1^2 = 2\int_0^{y_2} \varphi(t)f(y)dy,$$

whence as in the preceding case

$$\begin{cases} v_1^2 \leqq 2\varphi(t_2)F(y_2), \\ v_1^2 \geqq 2\varphi(t_1')F(y_2). \end{cases} \tag{5}$$

From (4) and (5) we get

$$F(y_1) \geqq F(y_2), \qquad \text{or} \qquad |y_1| \geqq |y_2|,$$
$$\varphi(t_1)F(y_1) \leqq \varphi(t_2)F(y_2).$$

A similar argument leads to the same results when y_1 is negative.

Starting now with the conditions $dy/dt = 0$, $y = y_2$, when $t = t_2$, we may repeat the entire argument and obtain $|y_2| \geqq |y_3|$, $\varphi(t_2)F(y_2) \leqq \varphi(t_3)F(y_3)$, and in general

$$|y_n| \geqq |y_{n+1}|, \tag{6}$$

and

$$\varphi(t_n)F(y_n) \leqq \varphi(t_{n+1})F(y_{n+1}), \tag{7}$$

where t_n is the nth value of t (beginning with t_1) for which $dy/dt = 0$, and y_n is the corresponding value of y.

The quantities y_n are the amplitudes of the successive oscillations. Hence (6) proves that the amplitudes decrease monotonically. From (7), together with the hypotheses regarding $\varphi(t)$, it may be shown that $F(y_n) \geqq (m/M)F(y_1)$, which proves that the amplitudes do not approach zero.

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A TWO-WAY INFINITE SERIES FOR LEBESGUE INTEGRALS*

BY M. B. PORTER

1. *The Defining Series.* Let $F(x)$ denote any positive measurable function of n variables defined over a measurable point set δ. We shall use the sign δ to denote either a point set or its measure, and the sign† δ^* to denote δ diminished by a nul-set or to indicate that a property holds *almost everywhere* (*presque partout*) over δ.

Now write

$$\Phi(\delta) = \sum_{i=-\infty}^{\infty} 2^i \phi_i(\delta) = \lim_{m=\infty} \sum_{i=-\infty}^{m} 2^i \phi_i(\delta), \tag{1}$$

where $\phi_i(\delta)$ is the measure of all those points E_i of δ for which $F(x)$, expressed in the binary scale, has unity in the ith place, it being understood that $F(x)$ shall be always expressed in closed form when possible (or else never so expressed), in order that the representation shall be unique. If $F(x)$ is limited, it is evident that (1) will converge uniformly over δ. If $F(x)$ is *not* limited, (1) may still converge; if it does, the convergence is necessarily uniform. In that case $F(x)$ is summable over δ, and we shall indicate the sum of the series by $\int F(x)$, the Lebesgue integral of F over δ. The integral is definite or indefinite according as we regard δ as fixed or as variable.

In case $F(x)$ is not always positive, set, as usual,

$\phi(x) = F(x)$ when $F(x) \geqq 0$, $\psi(x) = F(x)$ when $F(x) < 0$,
$\phi(x) = 0$ when $F(x) < 0$, $\psi(x) = 0$ when $F(x) \geqq 0$;

then

$$\Phi(\delta) = \sum_{-\infty}^{\infty} 2^i \phi_i(\delta) - \sum_{-\infty}^{\infty} 2^j \psi_j(\delta), \tag{2}$$

and this difference may converge even when the separate series diverge, if i and j become simultaneously infinite in a suitable manner.

* Presented to the Society, September 9, 1921.

† Cf. Pierpont's useful notation, *Functions of a Real Variable*, vol. 1, p. 119.

Thus, if

$$F(x) = \frac{d}{dx}\left(x \sin \frac{1}{x}\right), \qquad x \neq 0,$$

$F(x)$ is not summable at $x = 0$; but if from the interval (a, x), $a < 0$, $x > 0$, we remove the interval 2ϵ symmetric about the origin, the integral (2) will exist over the remaining domain and will be equal to $x \sin (1/x) - a \sin (1/a)$. A similar procedure can be applied where $F(x)$ has a countable set of non-summable points, and is the derivative of a continuous function.

Let η denote a set of infinitesimal hyperspheres about x, a point of δ, as center. We shall show that

$$\text{(3)} \qquad \operatorname*{Lim}_{\eta \to 0} \frac{\Phi(\eta)}{\eta}$$

exists over δ^*. Since this is a point-function we shall use the argument x in the limit.

Lebesgue has shown in his metric-density theorem that for the measure functions ϕ_i the limit (3) exists almost everywhere, and that†

$$\text{(4)} \qquad \begin{aligned} \phi_i'(x) &= 1 \quad \text{over} \quad E_i^*, \\ \phi_i'(x) &= 0 \quad \text{over} \quad C^*E_i, \end{aligned}$$

which in one dimension implies the existence of the derivative in the ordinary sense of the word over E_i^* and $C_i^*E_i$ with the values given in (4).

2. *Special Theorems.* From (4) we have the following theorems.

THEOREM A. *For any measurable function $F(x)$,*

$$F'(x) = \sum_{-\infty}^{\infty} 2^i \phi_i'(x) - \sum_{-\infty}^{\infty} 2^j \psi_j'(x),$$

both series converging over that part of δ^* where $F(x)$ is finite.

THEOREM B. *In case $F(x)$ is limited, $\int_\delta F(x)$ exists and its derivative is*

$$\int_\delta' F(x) = \sum_{-\infty}^{m} 2^i \phi'(x) - \sum_{-\infty}^{m} 2^j \psi'(x) = F(x)$$

over δ^*, the series converging uniformly over δ.

† Various proofs of this theorem have been given, e.g. a simple proof based on Vitali's covering theorem by de la Vallée Poussin, *Cours d'Analyse*, vol. 2, p. 110 et seq. See also Hobson, *Real Variables*, vol. 2, p. 178 et seq.

If $\psi(\delta)$ denote a function with a bounded upper symmetrical derived number $D^+\psi(x)$, then

$$f(\delta) = \int_\delta D^+\psi(x)$$

exists; and, since $f'(x) = D^+\psi(x)$ over δ^*, $f(\delta)$ and $\psi(\delta)$ differ* by a constant. In one dimension, this is Lebesgue's theorem that *a monotone increasing function with bounded derived numbers has a derivative almost everywhere.*

Theorem B can be extended by means of Vitali's covering theorem to any summable function, but we shall not stop to give here the details of the proof. The convergence of the series will be uniform over a set δ_1 differing from δ by as little as we please.

3. *Fundamental Theorems.* It will be readily seen that the following fundamental theorems follow at once from (1).

(1) The mean value theorem for integrals.

(2) The theorem that $\int_{\delta_1} f + \int_{\delta_2} f = \int_{\delta_1+\delta_2} f$.

where δ_1 and δ_2 have only a nul-set in common.

(3) The theorem that $\int_\delta [f_1 + f_2] = \int_\delta f_1 + \int_\delta f_2$,

proved by truncating f_1 and f_2, deriving, and then using Scheeffer's theorem.

(4) The theorem that $\operatorname{Lim}_{n\to\infty} \int_\delta f_n(x) = \int_\delta \operatorname{Lim}_{n\to\infty} f_n(x)$.

In conclusion we shall make some applications of Theorem A which will show its use in dealing with measurable functions.

4. *Application to Almost Everywhere Finite Measurable Functions.* If, in Theorem A, we cover each E_i set by a finite set of non-overlapping cells C_i such that

$$|\text{meas } E_i - \Sigma C_i| < \frac{\epsilon}{2^i},$$

and if we define a function equal to *one* over the C_i's and to *zero* over the complementary cells, we shall have a discontinuous function which can be made continuous by *trimming* the C_i cells or their complements and using suitable *connective tissue.* Thus we have the following theorem.

* In one dimension, this follows from Lebesgue's extension of Scheeffer's theorem, Lebesgue, *Leçons sur l'Intégration*, p. 79.

THEOREM C. *Any almost everywhere finite measurable function is equal to a continuous function save over a set of points of arbitrarily small measure* $\bar{\epsilon}$.*

If we denote this continuous function by $C(x)$, we have $\int_\delta F(x) = R\int_{\bar{\delta}} C(x) + \bar{\bar{\epsilon}}$ when δ and $\bar{\delta}$ differ by as little as we please, where $R\int_\delta$ denotes the Riemann integral.

5. *Egoroff's Theorem.* From § 4, we can deduce Egoroff's theorem:† *If a sequence of measurable functions* $F_i(x)$ *converges to a limit* $F(x)$ *over* δ *it will converge uniformly save over a subset of arbitrarily small measure.*

6. *Lusin's Extension of Weierstrass' Theorem.* In § 4, by using a set of $\bar{\epsilon}$'s converging to zero, we have the theorem: *Any in general finite measurable function is the limit almost everywhere of a sequence of continuous functions.* Hence we have also Lusin's‡ extension of Weierstrass' theorem, viz.: *Any measurable function is almost everywhere the limit of a sequence of polynomials.*

7. *Approximately Continuous Functions.* Making use of Denjoy's definition of *approximately continuous functions,*§ we have the theorem: *A measurable almost everywhere finite function is approximately continuous save for a nul-set.*

By means of Vitali's covering theorem it is easy to show that *any almost everywhere approximately continuous function is measurable.* Hence we can say that the *R*-integrable functions are almost everywhere continuous and the *L*-integrable functions are almost everywhere approximately continuous, when it is understood, of course, that the functions are limited.

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* Cf. Lusin, COMPTES RENDUS, June 17, 1912.

† COMPTES RENDUS, Jan. 30, 1911.

‡ Lusin, COMPTES RENDUS, loc. cit.

§ A function $F(x)$ is approximately continuous at x when it is continuous over a point set of metric density one at x. BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 43, p. 165; or Hobson, vol. 1, p. 295.

NOTE ON EULER'S φ-FUNCTION

BY R. D. CARMICHAEL

Two correspondents have recently called my attention to the fact that the supposed proof of the following theorem, which I gave some years ago,* is not adequate:

THEOREM I. *For a given number n, the equation $\varphi(x) = n$ either has no solution or it has at least two solutions.*

So far I have been unable to supply a proof of the theorem, though it seems probable that it is correct. I am therefore compelled to allow it to stand in the status of a conjectured or empirical theorem.

Let us examine the hypothesis that there exists a value ν of n such that $\varphi(x) = \nu$ has one and just one solution. It is easy to derive certain necessary properties of x. In the first place, x is even, since otherwise $2x$ would also be such that $\varphi(2x) = \nu$. Again, x is divisible by 4, since otherwise $\varphi(x/2)$ would be equal to ν. Let us then denote the value of x by $4s$. We shall prove the following theorem.

THEOREM II. *If $4s$ has the factor $p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_0\ (= 2)$, p_1, p_2, $\cdots$, p_k are distinct prime numbers, and if the quotient of $4s$ by this factor is prime to the factor, and if $p_0^{\gamma_0} p_1^{\gamma_1} \cdots p_k^{\gamma_k} + 1$ is a prime number q, where for a given i, $0 < \gamma_i < \alpha_i$, then $4s$ has the factor q^2.*

The proof is almost immediate. For we have

$$\varphi(2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = \varphi(2^{\alpha_0-\gamma_0} p_1^{\alpha_1-\gamma_1} p_2^{\alpha_2-\gamma_2} \cdots p_k^{\alpha_k-\gamma_k} q),$$

so that we should have two solutions of the equation $\varphi(4s) = \nu$ unless s contains the factor q. Similarly, it may be shown that s contains the factor q^2, since otherwise the first power of q could be omitted by appropriately raising certain (or all)

* This BULLETIN, vol. 13 (1907), p. 241. The theorem is also stated as an exercise in my *Theory of Numbers*, p. 36; it was its presence here that led each correspondent to the discovery of the error.

of the exponents on the p's without affecting the value of the φ-function.

In the same way we may prove the following theorem.

THEOREM III. *If s is divisible by a prime of the form $2^k + 1$ it is divisible by the square of this prime.*

For if $4s = 2^{\alpha_0}(2^k + 1)s_1$, where s_1 is prime to $2(2^k + 1)$, we have

$$\varphi(4s) = 2^{\alpha_0 - 1}2^k \cdot \varphi(s_1) = \varphi(2^{k+\alpha_0}s_1),$$

contrary to the hypothesis that the solution is unique.

From Theorem II, it follows that s has the factor 3^2, thence that it has the factor 7^2, and thence that it has the factor 43^2.

Now suppose that s does not have the factor 3^3. Then since $\varphi(2^{\alpha_0} \cdot 3^2) = 2^{\alpha_0} \cdot 3 = \varphi(2^{\alpha_0 - 1} \cdot 13)$, if $\alpha_0 > 1$, it is readily shown that s has the factor 13^2. For this case, then, $4s$ has the factor $2^2 \cdot 3^2 \cdot 7^2 \cdot 13^2 \cdot 43^2$. We may now apply Theorem II successively to show that $4s$ has the factors 79^2, 547^2, 3319^2, 1854763^2. It appears possible to determine in the same way still other factors of $4s$. Those obtained are sufficient to show that $4s$ has at least 38 digits.

If $4s$ contains the factor 3^3, then it follows from Theorem II that it has the factors 19^2, 127^2. Then $4s$ has the factor $2^2 \cdot 3^3 \cdot 7^2 \cdot 19^2 \cdot 43^2 \cdot 127^2$. Applying Theorem II successively we may show that $4s$ has the factors 2287^2, 101347^2, 304039^2. It appears possible to determine in the same way still other factors of $4s$. Thence it is easy to see that $4s$ in this case has at least 41 digits.

Hence, if the empirical Theorem I is not valid, the unique solution x (which would then exist for at least one value of n) must contain at least 38 digits. Moreover, it is easy to increase this number of necessary digits for these exceptional solutions. Hence it seems probable that the empirical theorem is true.

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EXTENSIONS OF DIRICHLET MULTIPLICATION AND DEDEKIND INVERSION*

BY E. T. BELL

1. *Introduction.* With Landau,† let us define *Dirichlet multiplication* to be the following formal process of combining two sequences α_n, β_n $(n = 1, 2, \cdots)$ to form a third sequence γ_n, $\gamma_n = \Sigma\alpha_l\beta_m$, the sum extending to all integers l, $m > 0$ such that $lm = n$. Henceforth, unless otherwise stated, we assume the four rational algebraic operations to be purely formal.‡ When any of these operations in relation to a given system of elements have special interpretations, examples of which are noted in a moment, they will be called *specific*. Restating Landau's definition, let us denote by α_n, β_n $(n = 1, 2, \cdots)$ two classes of elements which are such that the product of any element of one class by an element of the other has a unique significance, and likewise for any sum of such formal products. Then the *Dirichlet product* of these classes is the class $(\alpha, \beta)_n$ $(n = 1, 2, \cdots)$, where $(\alpha, \beta)_n = \Sigma\alpha_l\beta_m$ $(lm = n)$.

When the α's and β's are known, the process of determining the φ's from the relation $(\alpha, \varphi)_n = \beta_n$ for n an arbitrary integer > 0 will be called *Dedekind inversion*. It will be shown that the extension of this which we have in view includes Dedekind's inversion in the theory of numbers.§

The interpretations of the general theorems in a specific case will depend only upon the meanings assigned to the elements and to the rational operations upon those elements. Thus, if the elements are numbers and the rational operations are as in arithmetic, the interpretation is obvious; if the elements are classes, multiplication and addition are logical,

* Presented to the Society, October 22, 1921.

† Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 2, p. 671. We replace his series by sequences.

‡ As in the customary definitions of an abstract field, cf. Dickson, *Linear Groups*, pp. 5–6. When some of the operations in a field are specific, the field will be called special. See also § 6.

§ The inversion is usually attributed to Dedekind, although it was published simultaneously by Liouville. Cf. Dickson's *History*, vol.1, p. 441.

their inverses are not defined, and the theorems apply only to all sums and products derived from products which are themselves obtained by multiplications from a primary set;* if the elements are single-valued functions (or if they are associates, as presently defined), the important cases for the arithmetic of integers or of ideals, the interpretations of the rational operations are as developed later in this paper. By means of the last a great mass of the material which is summarized in Chapters V, X, XIX of volume I of Dickson's *History of the Theory of Numbers*, and which is not most naturally derived from the elements of elliptic functions, can be simultaneously restated in much condensed form, unified, proved almost at a glance, and extended in many directions. This, the most immediate application of what follows, will be discussed elsewhere.†

2. *Definitions.* (i) Let A' be a class of n' independent elements x_a ($a = 1, 2, \cdots, n'$) subject only to the commutative and associative laws of multiplication. By definition the zero powers of all elements in A' are equal, and each is equal to the multiplicative unit e, thus $x_a^0 = e$ ($a = 1, 2, \cdots, n'$), $e^2 = e$, etc.; and n' may be finite or infinite. From the elements of A' we form the class A of all products of the type $x_a^\alpha x_b^\beta \cdots x_c^\gamma \equiv \Pi x_a^\alpha$, where $a, b, \cdots, c$ are different members of the set $1, 2, \cdots, n'$, and $\alpha, \beta, \cdots, \gamma$ are integers $\geqq 0$, and denote this typical product by x. Products differing only by unit factors (powers of e) are equal. Hence if n' is finite, A may consist of either a finite or infinite number n of distinct products. The class A evidently includes all the members of A'. The members of A' are called the *primary elements* of A. Elements of A' are called the *derived elements*, and x is a typical derived element. From these definitions we may con-

* Subtraction and division are omitted for the reasons given by Whitehead, *Universal Algebra*, vol. 1, p. 82.

† Another application of importance for arithmetic, which has been developed in detail, is to the case in which the elements are sequences. A brief account of results for elliptic functions cited above which involve Bernoullian functions appeared recently in the MESSENGER OF MATHEMATICS; generalizations for certain of the other divisor theorems depending on elliptic functions will be published later.

sider the primary elements to be independent generators of an abelian group A whose identity is e. The order of the group is n, which may be finite or infinite. When A' is the class of prime numbers and multiplication is as in arithmetic, the resulting theory is called the numerical case.

(ii) An *associate* of an element in A is anything that has a unique significance in terms of elements of A, when the element, called the *argument* of the associate, is assigned. It is assumed that associates admit of unambiguous combination by formal rational operations (§ 1) with associates. If A be replaced throughout in what precedes by A', the class of primary elements of A, there is defined a primary associate. When the argument is a derived element, the associate is called derived. This distinction is made because all that follows is based upon associates that need exist only when their arguments are primary elements; derived associates are constructed (in (iv)) from the primaries.*

(iii) The notation being as in (i), the $\nu(x) = (\alpha + 1)(\beta + 1) \cdots (\gamma + 1)$ derived elements x' given by

$$x' = x_a^{\alpha'} x_b^{\beta'} \cdots x_c^{\gamma'} \quad (0 \leqq \alpha' \leqq \alpha,\ 0 \leqq \beta' \leqq \beta,\ \cdots,\ 0 \leqq \gamma' \leqq \gamma)$$

are by definition the divisors of the typical derived element x. In the product $x \equiv \Pi x_a^{\alpha}$ of powers of distinct primary elements, corresponding Latin and Greek letters (a, α), (b, β), $\cdots$, (c, γ) are associated with a given primary element, thus

* Since it seems not to be customary except in mathematical logic (*Principia Mathematica*, vol. 1, pp. 15–18) to speak of either values or functions apart from a numerical context, we use *associate* instead of *single-valued function*, although the latter would be justified by the current use of *propositional function*. For special interpretations of the several parts of (ii) the definition of associate degenerates as follows to that of single-valued function of an integer > 0. Let $\varphi(x)$ be an associate having the integral non-zero argument x. Give the parts of (ii) the following specific meanings, which obviously are legitimate: the rational operations are as in ordinary algebra; $A \equiv$ the class of integers > 0; the phrase "when the element is assigned" $\equiv$ *when x lies in the interval (a, b) of A;* "has a unique significance in terms of elements of A" $\equiv$ *has a single definite value corresponding to each x $(a \leqq x \leqq b)$, no matter in which form the correspondence is specified.* With these interpretations, $\varphi(x)$ is a single-valued function of x in the interval (a, b). Cf. Dirichlet's definition (*Werke*, vol. 1, p. 135), also Hobson, *Theory of Functions of a Real Variable*, 1st edition, p. 216) for the analogous definition when x, (a, b) are continuous.

$x_a^\alpha, x_b^\beta, \cdots, x_c^\gamma$. This convention is maintained throughout when the exponent is an arbitrary positive integer.

(iv) With each positive integral or zero power x_d^i of any primary element of x_d are associated specific primary associates $\varphi_i(x_d), \psi_i(x_d), \cdots, \mu_i(x_d), \cdots$ of that element, the suffix i of the primary associate being equal in each case to the exponent of the power. For example the φ-primary associates of x_a^0, $x_a^1, x_a^2, \cdots$ are $\varphi_0(x_a), \varphi_1(x_a), \varphi_2(x_a), \cdots$. By definition all primary associates of e are equal, and each is denoted by ϵ. Hence since by (i) $x_a^0 = x_b^0 = \cdots e$, we have $\varphi_0(x_a) = \varphi_0(x_b) = \cdots = \epsilon$; $\mu_0(x_a) = \mu_0(x_b) = \cdots = \epsilon$, etc. We call ϵ the *unit associate. It is assumed that, with respect to any associate, ϵ has all the multiplicative properties of unity in arithmetic.* Beyond the trivial one that they both have the same argument x_a, it is not assumed (without explicit statement of the assumption) that there is any relation whatever between $\varphi_m(x_a)$, and $\varphi_n(x_a)$ (m, n integers $\geqq 0$, $m \neq n$); nor is it assumed that primary associates having different suffixes are necessarily distinct.*

From the definition in (ii), it follows that, for $x = \Pi x_a^\alpha$ as in (i),

$$\Pi\varphi_\alpha(x_a) \equiv \varphi_\alpha(x_a)\varphi_\beta(x_b) \cdots \varphi_\gamma(x_c)$$

is an associate of x. We call this a *derived associate*, and we shall denote it by φ_x. The derived associate $\varphi_{x'}$ of the divisor x' of x is called an *associate divisor* of φ_x. Similarly, starting with the primary associates $\psi_\alpha(x_a), \psi_\beta(x_b), \cdots, \psi_\gamma(x_c)$, we define $1 + \nu(x)$ derived associates $\psi_x, \psi_{x'}$; and so on for $\chi_x, \chi_{x'}$; $\lambda_x, \lambda_{x'}$; $\cdots$; $\mu_x, \mu_{x'}$. Thus $\mu_x = \Pi\mu_\alpha(x_a)$. Henceforth we shall drop the adjectives *primary* and *derived*, and refer only to associates, since the notations $\varphi_a(x_a)$, φ_x suffice without qualifications.

3. *Relation with Group Characters.* Before proceeding, we emphasize the remarks made in § 1 regarding the interpretations of results. This has particular reference to the assump-

* For example in the special case of associate = single-valued function, we may have $\varphi_\alpha(x_a) = \sin \pi x_a$, $\varphi_\beta(x_b) = \sin \pi x_b$; namely each is the sine of π times its argument.

tion in § 2 (ii), to which we shall add some comments in § 6. The development is necessarily abstract, because in the applications it is essential that proofs be based upon considerations that are independent of convergence, or indeed of any infinite process, and because each general theorem is susceptible of many specific interpretations. (Cf. § 1.) It may assist the reader to consult § 8 occasionally. We insist, however, that nothing in § 8 is assumed in any of the proofs, all of which are immediate consequences of the definitions, except that those after Theorem XIV depend also upon the assumption that an arithmetic system exists. This assumption is satisfied by the class of all rational functions in any number n of independent variables with integral coefficients, which for $n = 1, 2$ is the important case arithmetically. We have assumed that for each primary element there exists at least one associate. This can be satisfied, for example, by taking the associate equal to the element. Hence the theory as developed relates to objects that exist.

In addition to possible interpretations mentioned in § 1 we note one here which is of particular interest.* If we impose the restrictions $f_2(x_a) = [f_1(x_a)]^2$ ($f = \varphi, \psi, \chi, \cdots, \lambda, \mu, \cdots$) upon the associates, they can be interpreted as group characters.†

4. *Extended Dirichlet Product.* If the derived elements x_1, x_2 (§ 2 (i)) have in common only unit factors (powers of e), they are called *relatively distinct* (the analogue of *relatively prime*).

THEOREM I. *If x_1, x_2 are relatively distinct, $\varphi_{x_1}\varphi_{x_2} = \varphi_{x_1x_2}$.*

Next, let x_1, x_2 be *any* pair of divisors (§ 2 (iii)) of x that are such that $x_1x_2 = x$; form $\varphi_{x_1}\psi_{x_2}$ for each pair (x_1, x_2); take the sum of all such products for all possible pairs (x_1, x_2); and denote that sum by $(\varphi, \psi)_x$. It is obvious, by § 2 (ii), (iv), that this sum is an associate of x, and hence the process

* This was pointed out by Professor Dickson. I had not noticed it, since I was developing the theory with reference to its applications to sequences, classes, and the multiplicative properties of numbers and ideals.

† For the case of abelian groups, cf. Weber, *Algebra*, 2d edition, vol. 2, p. 43; for an exposition of the general case, cf. Dickson, ANNALS OF MATHEMATICS, (2), vol. 4, 1902, pp. 25–49.

may be repeated for $(\varphi, \psi)_x$ and χ_x as initial associates instead of φ_x, ψ_x; denote the result by $((\varphi, \psi), \chi)_x$, or briefly by $(\varphi, \psi, \chi)_x$. Continuing thus, starting with m associates $\varphi_x, \psi_x, \chi_x, \cdots, \lambda_x, \mu_x$, taken in this order, we reach finally an associate $(\varphi, \psi, \chi, \cdots, \lambda, \mu)_x$ of x, which is called the *Dirichlet product* of the original m associates, and the original m associates are called the *primitive divisors* of the extended product.

THEOREM II. *The extended product* $(\varphi, \psi, \cdots, \mu)_x$ *is a symmetric function of* $\varphi, \psi, \cdots, \mu$. *In other words, extended multiplication is associative and commutative.*

This being a result of great power in applications, we shall examine it more closely. Let $x_1, x_2, \cdots, x_m$ denote divisors of x such that $x = x_1 x_2 \cdots x_m$. Then we have

$$(\varphi, \psi, \cdots, \mu)_x = \Sigma \varphi_{x_1} \psi_{x_2} \cdots \mu_{x_m},$$

where the sum extends to all sets $(x_1, x_2, \cdots, x_m)$. The m functions in the typical term on the right are associate divisors (§ 2 (iv)) of $\varphi_x, \psi_x, \cdots, \mu_x$, respectively. From the mode of formation it is evident that the sum (i.e., the extended Dirichlet product) is invariant under permutations of some or all of $x_1, x_2, x_3, \cdots, x_m$. This sum is an associate of x, say $F_x^{(m)}$, which is uniquely determined whenever x and the primitive divisors $\varphi_x, \psi_x, \cdots, \mu_x$ of $(\varphi, \psi, \cdots, \mu)_x \equiv F_x^{(m)}$ are assigned; in particular it is uniquely known when x is replaced by any one of its divisors x'. In the same way, starting with any n associates $\alpha_x, \beta_x, \cdots, \gamma_x$, we arrive at $G_x^{(n)}$, and continuing thus we obtain a system of r such extended products $F_x^{(m)}, G_x^{(n)}, \cdots, H_x^{(t)}$, there being r letters $m, n, \cdots, t$. Let $m + n + \cdots + t = s$, and suppose that the s associates upon which the r extended products are constructed are $\alpha_x, \beta_x, \cdots, \omega_x$, these being not necessarily distinct. Then we have

$$(F^{(m)}, G^{(n)}, \cdots, H^{(t)})_x = (\alpha, \beta, \cdots, \omega)_x.$$

Except when $m = n = \cdots = t = 1$, the primitive divisors $F_x^{(m)}, G_x^{(n)}, \cdots, H_x^{(t)}$ and $\alpha_x, \beta_x, \cdots, \omega_x$ of these extended products are not the same. On the left, the extended product refers to resolutions of x into r divisors which in turn are resolved into $m, n, \cdots, t$ divisors respectively, the resolutions in

both cases being all possible of their respective kinds; on the right, the resolution is into all possible sets of s divisors. The identity of the formal sums for these distinct methods of resolving x is the interpretation of the associative and commutative properties of extended multiplication. The significance of Theorem II in the numerical case was stated in a former paper.*

In order to see in the numerical case that $(\varphi, \psi)_x$ is equivalent to $(\alpha, \beta)_n$ of § 1 when φ_x, ψ_x, and the α's and β's are single-valued functions of integers, take as primary elements the natural primes, choose $\varphi_\rho(x_r) = \varphi'(x_r{}^\rho)$ where φ' is single-valued and such that $\varphi'(mn) = \varphi'(m)\varphi'(n)$ when m and n are relatively prime integers greater than zero. In the same way, define $\psi_\rho(x_r) = \psi'(x_r{}^\rho)$, and finally write $\varphi'(x)$, $\psi'(x) \equiv \varphi_x'$, ψ_x'. Then for α, β, n in the numerical case equal respectively φ', ψ', x, as above, we have $(\varphi', \psi')_x = (\alpha, \beta)_n$.

Next, the sum of the associates $\chi_x, \cdots, \lambda_x, \mu_x$ is obviously an associate, σ_x, of x, and it is easily seen that

$$(\varphi, \sigma)_x = (\varphi, \chi)_x + \cdots + (\varphi, \lambda)_x + (\varphi, \mu)_x.$$

THEOREM III. *Extended Dirichlet multiplication is distributive with respect to (formal) addition.*

5. *Extended Dedekind Inversion.* The unit associate ϵ was defined in § 2 (iv). We now introduce the *associate unit* $\eta_x \equiv \eta$, whose definition is as follows: $\eta_x = 0$ (the multiplicative zero) when $x \neq e$ (the multiplicative unit of A); $\eta_x = \epsilon$ (the unit associate) when $x = e$.

THEOREM IV. *For each associate φ_x there exists a unique associate φ_x' such that $(\varphi, \varphi')_x = \eta$.*

We shall call φ_x' the *reciprocal* of φ_x, and, by an obvious algebraic analogy, we shall write $\varphi_x' = \eta/\varphi_x$, $\varphi_x = \eta/\varphi_x'$: We have at once, ψ_x being any associate, $(\psi, \eta)_x = \psi_x$; and hence we have the following theorem.

THEOREM V. *The unit with respect to extended Dirichlet multiplication is η.*

To prove Theorem IV, we shall exhibit φ_x' explicitly in terms

* This BULLETIN, vol. 27, p. 274, § 2.

of φ-associates. We observe first that if $x = \Pi x_a^{\alpha}$ as usual, Theorem I enables us to state that φ_x' is equal to the formal product of the associates φ_ξ' for $\xi = x_a^{\alpha}, x_b^{\beta}, \cdots, x_c^{\gamma}$. Hence it is sufficient to determine φ_x' when $x = x_a^{\alpha}$, $\alpha > 0$. By the convention of § 2 (iv), the φ-primary associates of $x_a^1, x_a^2, \cdots, x_a^{\alpha}$ are respectively $\varphi_1(x_a), \varphi_2(x_a), \cdots, \varphi_\alpha(x_a)$. For $x = x_a^{\alpha}$, write φ_x' in the form $\varphi_\alpha'(x_a)$. Then it can be verified without difficulty that $\varphi_\alpha'(x_x)$ is equal to the determinant

$$\left(-\frac{1}{\epsilon}\right)^{\alpha} \begin{vmatrix} \varphi_1(x_a) & \epsilon & 0 & \cdots & 0 \\ \varphi_2(x_a) & \varphi_1(x_a) & \epsilon & \cdots & 0 \\ \varphi_3(x_a) & \varphi_2(x_a) & \varphi_1(x_a) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_{\alpha-1}(x_a) & \varphi_{\alpha-2}(x_a) & \varphi_{\alpha-3}(x_a) & \cdots & \epsilon \\ \varphi_\alpha(x_a) & \varphi_{\alpha-1}(x_a) & \varphi_{\alpha-2}(x_a) & \cdots & \varphi_1(x_a) \end{vmatrix}.$$

It is easy to prove this also by mathematical induction from $\alpha - 1$ to α, $\alpha > 1$. Next, φ_x' is written down as the formal product of $\varphi_\alpha'(x_a)$, $\varphi_\beta'(x_b)$, $\cdots$, $\varphi_\gamma'(x_c)$, which are obtained from the above on replacing (α, a) in turn by (α, a), (β, b), $\cdots$, (γ, c). It may be mentioned that in practice the explicit form of φ_x' is not often required; its existence is the important fact. Henceforth we indicate the reciprocal of a given associate by accenting the symbol of the latter as above.

THEOREM VI. *There is a unique φ_x satisfying the relation $(\varphi, \mu)_x = \omega_x$ when μ_x, ω_x are given: $\varphi_x = (\omega, \mu')_x$.*

For, from the given relation, $(\omega, \mu')_x = ((\varphi, \mu), \mu')_x = (\varphi, \mu, \mu')_x = (\varphi, (\mu, \mu'))_x = (\varphi, \eta)_x = \varphi_x$. In the same way, or as a corollary, we have the general case of such inversion:

THEOREM VII. *If $(\varphi, \psi, \cdots, \lambda, \chi, \sigma, \cdots, \mu)_x = (\pi, \tau, \cdots, \omega)_x$, then $(\chi, \sigma, \cdots, \mu)_x = (\pi, \tau, \cdots, \omega, \varphi', \psi', \cdots, \lambda')_x$.*

An interpretation in the numerical case of Theorem VI is as given in § 3 of the paper cited in § 4. For let the associates be single-valued functions of their arguments. Then if $\varphi_i(n) = 1$ for $n > 0$ an integer in the formula $\varphi_i = \varphi_k\varphi_i'$ stated in that paper, $\varphi_i'(n)$ is Möbius' $\mu(n)$, and the result is Dedekind's inversion, which thus appears as a very special consequence of the theorem in the numerical case.

6. *The Special Field of all Associates.* For greater clearness, before stating the next theorem, we recall the significance of the terms involved. For a rigorous discussion of the relevant logic we must refer the reader to *Principia Mathematica*, volume 1, pages 15–27, 85–88, 143–156, 260–268, 278–291. The term *real variable* is used as defined on page 18 of that work. The elements of an abstract field are real variables; theorems relating to an abstract field are assertions of propositional functions (ibid., pp. 15, 19). Our fundamental assumption in § 2 (ii) is "associates admit of unambiguous combination by formal rational operations with associates." Thus, in particular, it is assumed that in the abstract field φ_ξ (ξ = any element of A, φ_ξ = any associate of ξ), statements of identity between formal sums, differences, products, and quotients of elements (associates) in the field are propositional functions. On referring to the definitions of *extended Dirichlet product* and *reciprocal*, we see that only formal sums and products of associates are involved in the former, and only formal division by a power of ϵ, formal sums, differences and products of associates are involved in the latter. Hence statements of identity between Dirichlet products and reciprocals are propositional functions; and Theorem VIII is (*Principia Mathematica*, p. 18) an "ambiguous assertion" for every such propositional function in the abstract field of associates. These propositional functions in every case, *when Dirichlet products and reciprocals are written in full*, reduce to identities in *formal* addition (+), subtraction (−), multiplication (ab, etc.), division (a/b), when, as assumed, these operations have with respect to associates $a, b, \cdots$ their usual formal properties, $a + a - a = a$, $ab = ba$, $a(b + c) = ab + ac$, etc.

When in Theorem VIII special meanings are assigned to the associates and operations, it becomes a proposition concerning a special class of associates. An important case is that in which the associates are single-valued functions. To formal addition in this case we give the following meaning: The elements x are those of A, § 2 (i). If φ_x has a unique value

$\bar{\varphi}_x$ for each element x, and ψ_x a unique value $\bar{\psi}_x$ for each x, then the single-valued function having for each element x the value $\bar{\varphi}_x + \bar{\psi}_x$ is called the sum $\varphi_x + \psi_x$ of φ_x and ψ_x. Similarly for formal subtraction. For formal multiplication in this case, the single-valued function having for each element x the value $\bar{\varphi}_x\bar{\psi}_x$ is called the *product* $\varphi_x\psi_x$ of φ_x and ψ_x; and similarly for division. And so in each special case of Theorem VIII, the indicated *specific* operations of Dirichlet multiplication as defined in § 4 and division (extended Dedekind inversion) as in Theorem VII, may be said to have been broken down into their equivalents in terms of formal operations, to which are given special interpretations according to the special meanings assigned to the associates. Combining Theorems II, III, V, VII we have now the following powerful result.

THEOREM VIII. *All associates of the elements of an abelian group constitute a special field in which addition and subtraction are formal, multiplication is extended Dirichlet multiplication and division is extended Dedekind inversion.*

7. *Extended Cauchy Product.* Although Theorem VIII is sufficient for most purposes, it is of interest in many applications to proceed from it to an arithmetic classification of the results. This is not always feasible, but a majority of the known theorems on the arithmetic functions mentioned in the introduction, and the entire body of results of which these are but a part, can be so classified upon the following simple basis.

Restating Landau's definition (loc. cit., p. 670) of Cauchy multiplication, we call the class $[\alpha, \beta]_s$ $(s = 0, 1, \cdots)$ the *Cauchy product* of the classes α_s, β_s $(s = 0, 1, \cdots)$, when $[\alpha, \beta]_s = \sum_{r=0}^{s} \alpha_r\beta_{s-r}$. Repeating the process for $[\alpha, \beta]_s$ and γ_s $(s = 0, 1, \cdots)$, we form the class $[\alpha, \beta, \gamma]_s$ $(s = 0, 1, \cdots)$, and so on, getting finally, for any number of classes α_s, β_s, $\cdots$, δ_s $(s = 0, 1, \cdots)$, the Cauchy product of all of those classes, $[\alpha, \beta, \cdots, \delta]_s$ $(s = 0, 1, \cdots)$. We call α_s, β_s, $\cdots$, δ_s the factors of $[\alpha, \beta, \cdots, \delta]_s$. Clearly $[\alpha, \beta, \cdots, \delta]_s$ is invariant under permutation of some or all of its factors (this is equiva-

lent to permuting $\alpha, \beta, \cdots, \delta$ within $[\alpha, \beta, \cdots, \delta]_s$); and if $\alpha_s + \beta_s = \gamma_s$ $(s = 0, 1, \cdots)$, then $[\gamma, \delta]_s = [\alpha, \delta]_s + [\beta, \delta]_s$. That is, this multiplication is distributive.

THEOREM IX. *Extended Cauchy multiplication is associative, commutative, and with respect to formal addition, distributive.*

The important case arithmetically is that in which the factors of a Cauchy product are single-valued functions. Let $x = x_a^{\alpha} x_b^{\beta} \cdots x_c^{\gamma}$ as in preceding sections. Then the following theorem is an important consequence of the definitions.

THEOREM X. *The sign* Π *of formal multiplication extending to all* $(\rho, r) = (\alpha, a), (\beta, b), \cdots, (\gamma, c)$,

$$(\varphi, \psi)_x = \Pi[\varphi(x_r), \psi(x_r)]_\rho \equiv \Pi[\sum_{s=0}^{\rho} \varphi_s(x_r)\psi_{\rho-s}(x_r)].$$

By repetition, we reach the following general result:

THEOREM XI.

$$(\varphi, \psi, \cdots, \mu)_x = \Pi[\varphi(x_r), \psi(x_r), \cdots, \mu(x_r)]_\rho.$$

From Theorems IV and X, we arrive at the inverse process, which we shall call *extended Cauchy division*, in the following theorems.

THEOREM XII. $[\varphi(x_a), \varphi'(x_a)]_a = \eta$;

THEOREM XIII. *The first of the following relations implies the second,*

$$[\varphi(x_a), \cdots, \lambda(x_a), \chi(x_a), \cdots, \mu(x_a)]_a = [\pi(x_a), \cdots, \omega(x_a)]_a,$$
$$[\chi(x_a), \cdots, \mu(x_a)] = [\pi(x_a), \cdots, \omega(x_a), \varphi'(x_a), \cdots, \lambda'(x_a)]_a.$$

Theorem XIII is analogous to Theorem VII, from which, in conjunction with Theorem X for $x \equiv x_a^{\alpha}$, it can easily be proved. Combining these results we have the analogue of Theorem VIII.

THEOREM XIV. *All the associates of the elements of an abelian group constitute a special field in which addition and subtraction are formal, while multiplication and division are extended Cauchy multiplication and its inverse.*

We shall call an abelian group of finite or infinite order which has a unique basis,* *an arithmetic system.* This name

* Cf. Weber, loc. cit., p. 33.

is chosen because such a group has the characteristic distinguishing arithmetic from algebra, that is, a unique factorization law. Arithmetic systems are abstract or special; in the former multiplication is abstract, in the latter, specific. Let us assume as a hypothesis that the class of all associates is an arithmetic system under extended Cauchy multiplication. Let us call the elements of the basis the *prime associates of the system:* a prime associate is the extended Cauchy product of no pair of prime associates each distinct from ϵ, η. From Theorem XI, we have then the following theorem.

THEOREM XV. *The specific arithmetic system of associates in which multiplication is extended Cauchy multiplication is also a specific arithmetic system in which multiplication is extended Dirichlet multiplication.*

8. *Connection with Formal Expansions.* With each element x_i $(i = 0, 1, \cdots)$ of an abelian group (or of an arithmetic system) let us associate an umbra z_i, and let us write

$$\overline{\varphi}(x, z) = \Pi_i[\varphi_0(x_i)z_i^0 + \varphi_1(x_i)z_i^1 + \cdots + \varphi_n(x_i)z_i^n + \cdots]$$

and similarly for $\overline{\psi}(x, z), \cdots, \overline{\mu}(x, z)$. Then the coefficients of $z_a^\alpha z_b^\beta \cdots z_c^\gamma$ in the formal developments of $\overline{\varphi}(x, z)$, $\overline{\varphi}(x, z)\overline{\psi}(x, z) \cdots \overline{\mu}(x, z)$, and the reciprocal of the latter, are, respectively, φ_x, $(\varphi, \psi, \cdots, \mu)_x$, $\eta/(\varphi, \psi, \cdots, \mu)_x$.

9. *Note.* We have nowhere intended to imply that the processes of this paper are more than *extensions* of Dirichlet multiplication and of Dedekind inversion. In fact it can be shown that the processes of this paper and those current in the theory of numbers are formally equivalent in the sense that either set may be inferred from the other. The forms in this paper are so stated that they provide powerful algorithms, in which all discussions of convergence are obviated by establishing identities between arithmetic functions. The theory can be generalized to associates $\varphi_{x, y, z, \ldots}$ of any number of independent classes (or abelian groups) of elements, but the generalization has no apparent interpretation in terms of ordinary numbers.

THE UNIVERSITY OF WASHINGTON.

CHARLES LEONARD BOUTON*

A MINUTE READ BEFORE THE FACULTY OF HARVARD UNIVERSITY
March 28, 1922

Charles Leonard Bouton was born in St. Louis, Missouri, April 25, 1869. His father, William Bouton, was of Huguenot descent, and the family was long established in New England. At the close of the Civil War, William Bouton settled in St. Louis, where his regiment had been disbanded. Charles's mother, Mary Rothery Conklin, was also of old American stock; her grandparents were Scotch. William Bouton was an engineer by profession. His grandfather is said to have been the projector of the Erie Railroad, and was the author of the first article on its construction. Charles was the only one of the four sons who did not follow in his father's footsteps. The home atmosphere was academic and intellectually stimulating.

Bouton received his early education in the public schools of St. Louis, and took his first degree, that of Master of Science, at Washington University in 1891. Here, he came under the instruction of a highly skilled teacher of descriptive geometry, Dr. Edmund Arthur Engler. The next two years were given to teaching in Smith Academy, St. Louis, and these were followed by a year as instructor in Washington University, part of his work being to assist Professor Henry S. Pritchett. His next, and as it turned out, his last move was to Harvard. The years '94–'95 and '95–'96 were spent in the Graduate School. He took the master's degree at the end of the first year, and at the end of the second he was awarded a Parker Fellowship for study abroad. His two years at Leipzig were most profitably spent. He chose as his master that most original geometer, Sophus Lie, then at the height of his fame. As a matter of fact, Bouton was one of the great Norwegian's last pupils, for Lie returned to Norway in 1898 and died soon after. All of Bouton's subsequent scientific work bore the clear impress of Lie's genius. His two advanced courses, which he originated soon after his return to Harvard, dealt respectively with the theory of geometrical transformations and the application of transformation groups to the solution of differential equations. The graduate students who subsequently had the good fortune to prepare for the doctorate under his care generally took up subjects connected with the theory of transformations.

After receiving the doctorate at Leipzig in 1898 Bouton returned to Harvard and began a long period of work, broken only by occasional sabbatical absence. He threw himself with the greatest zeal into his duties as a teacher. At one time or another, beside the alternating advanced courses mentioned, he taught nearly every one of the lower and

* Professor Charles Leonard Bouton died on February 20, 1922. See this BULLETIN, vol. 28, p. 82 (Jan.–Feb., 1922).

middle group courses in mathematics. No pains were too great for him to spend, either on the preparation of lectures or on helping the individual student, whether a Freshman or a candidate for the doctor's degree. His characteristic quality of scientific sanity was invaluable, for it led him always to emphasize that which was permanently important, and to avoid tinsel and sham. A fine example of his didactic sense is seen in a collection of problems on the construction of Riemann's surfaces, published in volume 12 (1898) of the ANNALS OF MATHEMATICS. He was equally successful in arousing the interest of a beginner by showing him a model or a diagram or an enlightening example of a new theory, and in guiding a graduate with sure hand toward researches of permanent value and importance.

Those qualities which made Bouton an admirable teacher were conspicuous in his other professional activities. He was an editor of the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY from 1900 to 1902, and an associate editor of the TRANSACTIONS of the same society from 1902 to 1911. His power of keen yet kindly criticism, and his unerring mathematical judgment made him an efficient referee. His advice was prized by all who knew him, his opinion was always heard with respect, and his sanity was no less remarkable than his unselfishness. All of these qualities were drawn upon in full measure in the autumn of 1918 when, almost overnight, he was called to organize the mathematical instruction of nearly a thousand men in the Students' Army Training Corps. He carried this work through with conspicuous success, and the leading teachers of mathematics in the schools of this community, who enthusiastically rallied to the support of Harvard and the nation in that crisis, found in him a helpful guide and an efficient administrator.

His home life was beautifully quiet and peaceful. In 1907 he married Mary Spencer of Baltimore, and she, with their three daughters, Elizabeth, Margaret, and Charlotte, survives him. Yet for some time before the end, long dark shadows were crossing his life. The persistent after-effects of a hurried operation for appendicitis seemed to sap his strength. Family cares and anxieties multiplied, reaching a crisis in 1918 with the death of his youngest child. His breakdown in 1921 seemed but the inevitable end toward which events had long been tending. His death deprived the university of a faithful servant, and the community of a single-minded and upright gentleman.

WILLIAM F. OSGOOD
(Signed) JULIAN L. COOLIDGE
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KLEIN'S COLLECTED WORKS

Felix Klein: Gesammelte mathematische Abhandlungen. Vol. 1. *Liniengeometrie, Grundlegung der Geometrie, Zum Erlanger Programm.* Edited by R. Fricke and A. Ostrowski. Berlin, Julius Springer, 1921. xiii + 612 pages.

Felix Klein, born at Düsseldorf, April 25, 1849, was awarded the doctor's degree December 12, 1868, by the University of Bonn. The dissertation was directed and the examination conducted by Professor Lipschitz. At the golden jubilee of the doctorate, celebrated at Göttingen, December 12, 1918, plans were perfected for the publication of the numerous and varied mathematical contributions he made during the ensuing fifty years.

This was a particularly fitting thing to do. During this time no one exercised a greater influence on the development of mathematics in Germany than he. That that influence was not confined to Germany is attested by the long list of Americans who have taken the doctorate under his direction, and the very large number who have come from all parts of the world to hear his lectures or to participate in his Seminar. Notwithstanding the excellence of his personal contributions, and the inspiration of personal contact with him, probably his strongest characteristic was his skill as an organizer, and his capacity for cooperation.

Klein is essentially a geometer, and he began his work just in the zenith of the teaching of Clebsch, Kummer, Plücker and von Staudt; a worthy member of the group which included Lie, Darboux, Cremona, Clifford, Noether . . . How important are their achievements in the study of the more critical concepts of number, integral, limit, point set, etc., of to-day!

Of the three sub-titles indicated, the memoirs on line geometry occupy the first 240 pages. While the text is nowhere materially changed, a few corrections are introduced, indicated by square brackets, a considerable number of comments so marked are found in the footnotes, and each longer memoir or group of related memoirs is prefaced by biographical material written by the author, which add not only to the human interest, but aid in the understanding of the relations between various questions considered.

While at Bonn, Klein was Plücker's assistant, not only in the preparation of lecture notes on theoretical physics, but also in the writing of the *Line Geometry.** Klein intended to make physics his chief study, but acquired so much momentum in this work that he could not stop. At the time of Plücker's death the first part of the *Line Geometry* was practically all in print, but much of the second part was undeveloped. Plücker had started Klein on the problem of the quadratic complex, and the supervision of his progress was now taken over by Lipschitz. Plücker's treatment of the new idea was detailed and elementary, strikingly different from the style of Battaglini, who wrote on the quadratic complex shortly after Plücker's

* *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement.* Part I, pp. 1–226, edited by Clebsch, 1868; Part II, pp. 227–378, edited by Klein, 1869.

earlier papers had appeared.* Klein discovered that Battaglini's form was not the most general one, and presented the manuscript of a thesis in which the complex and the quadratic identity are both expressed as the sum of squares. But in the fall of 1868 Lipschitz received the proof-sheets of Weierstrass's forthcoming paper on elementary divisors, and he asked the candidate to include the various cases in his dissertation. The principles of the classification are found in the dissertation but the details of the transformation and its geometric meaning are reserved for one of Klein's own pupils later† and the numerous inaccuracies found in the interpretation have been corrected by various Italian mathematicians. As soon as his examination was completed, Klein went to Göttingen to be with Clebsch while preparing the second part of the *Line Geometry*, and remained until the end of the following summer semester. Both Clebsch and Noether were working on the problem of mapping an algebraic surface on the plane.

The following winter semester Klein spent in Berlin; there he met L. Kiepert, O. Stolz, and S. Lie. He heard no lectures, but took an active part in the Seminar conducted by Kummer and Weierstrass. Stolz had told Klein about non-euclidean geometry, and the latter soon felt that it would furnish an interpretation of Cayley's absolute measurement, which had already made a profound impression upon him. Curiously, Weierstrass emphatically rejected the whole idea, as distance was too invariant to be subjected to change of scale. In the spring, both Klein and Lie went to Paris, where they lived in adjoining rooms until the war broke out in July. Klein reported at once for duty, but was rejected on account of frail health; later he participated in relief work until he was taken ill with typhoid fever, which kept him bedfast for several weeks. Lie visited him while he was still an invalid, and together they wrote the paper on the asymptotic lines of the Kummer surface. Klein returned to Göttingen as a docent in January, 1871. He remained there until the call to Erlangen, in 1872. In the summer of 1871, Stolz also came to Göttingen.

With the exception of two titles (numbers XII, XIII) all of the papers appearing in this first part were written during these three years. These later papers were also concerned with line geometry, but the methods employed in them are essentially transcendental. In the preface it is explained how they naturally fit into the earlier category. They are *Ueber Konfigurationen, welche der Kummerschen Fläche zugleich eingeschrieben und umgeschrieben sind*, (MATHEMATISCHE ANNALEN, vol. 27 (1885)) and *Zur geometrischen Deutung des Abelschen Theorems der hyperelliptischen Integrale* (MATHEMATISCHE ANNALEN, vol. 28 (1886)). They were suggested by the dissertations of some of Klein's students, particularly of Rohn‡ and of Domsch.§ The most important contribution of Klein to

* *Intorno ai sistemi di rette di secondo grado*, ATTI DELLA ACCADEMIA DI NAPOLI, vol. 3 (1866).

† A. Weiler, *Ueber die verschiedenen Gattungen der Complexe zweiten Grades*, MATHEMATISCHE ANNALEN, vol. 7 (1872).

‡ *Betrachtungen über die Kummersche Fläche und ihren Zusammenhang mit den hyperelliptischen Functionen* $p = 2$, Munich, 1878.

§ *Die Darstellung der Fläche vierter Ordnung mit Doppelkegelschnitt durch hyperelliptische Functionen*, Leipzig, 1886.

line geometry is his system of coordinates, and its application to systems of quadratic complexes, to confocal systems, and to differential properties.

The second part of the present volume, pages 240–410, contains a number of early notes, and the long memoir of 1890, which contains a substantial harmonization of the earlier notes on measurement, the theory of groups, and the significance of transformations which leave a given configuration —space, variety, surface, curve—invariant. Although von Staudt's *Geometrie der Lage* was written in 1847, it was not noticed until 1871 that his treatment of harmonic elements, upon which the later *Beiträge* built the theory of coordinates, was essentially independent of the parallel axiom; it was Klein who made it sharply and distinctly so. The first paper is a brief summary which first appeared in the GÖTTINGER NACHRICHTEN, showing that Cayley's theory of measurement can be forcefully visualized by means of von Staudt's projective geometry. Besides these two sources the author calls attention to the help he got from the memoir of Beltrami* and particularly to the great inspiration which he found in the writings of Clifford. He states that he found the books of Salmon most useful and instructive.

In the fall of 1873 Klein attended the Bradford meeting of the British Association, and met Cayley, Sylvester, Clifford, and Ball (Sir Robert). The two long papers *Ueber die sogenannte Nicht-Euklidische Geometrie* had already appeared before the Bradford meeting was held, as the second was finished in 1872. The next paper was written after the Bradford meeting. It contains the axiom of projectivity in its present form, and also the statement that continuity can be postulated in the same manner as in metric geometry. Now follows a gap of six years, at the end of which appears the short note in which the author acknowledges the redundancy of his system of axioms necessary for the von Staudt development; it is a further analysis of the axiom of projectivity. The last paper appeared in 1890, at the end of the course on non-euclidean geometry given at Göttingen in 1889–1890. It contains a detailed account of Clifford's theory of parallels (right and left parallels), points out the difference between spherical and elliptic geometry, develops a purely projective foundation for analytic geometry, and discusses the meaning of axioms in general.

The third volume of Lie's *Theorie der Transformationengruppen* appeared in 1893. The preface contains (pp. x–xii) an admirable resumé of the growth of non-euclidean geometry and of the significance of the problem discussed in Part V, pages 393–543. It also contains several statements regarding the contributions of Klein to the development of non-euclidean geometry and of the foundations of geometry that are sadly out of keeping with the worth and the dignity of the other parts of that monumental work. In 1897 Klein was invited to report on the value of Part V of Lie's volume 3 to the Physico-Mathematical Society of the imperial University of Kasan in connection with the first award of the Lobachschevsky prize. The report covers 18 pages in the present volume. A more dignified or noble eulogy can hardly be found in critical literature than that sent by Klein to the Russian Society, proposing Lie's name. Revenge is sweet.

* *Saggio di interpretazione della geometria non euclidea*, GIORNALE DI BATTAGLINI, vol. 6 (1868).

The third part is concerned with the Erlangen Programm, but there is no sharp division between this and the preceding one. It begins with the joint note on W-curves, written by Klein and Lie, first the summary that appeared in the COMPTES RENDUS, then the full paper that was published a little later in the ANNALEN. In this paper one sees the different points of view of the two authors, one striving to express the phenomena as discrete operations, the other to define everything in terms of differential equations. It is a perfectly natural suggestion growing out of their joint work on non-euclidean geometry. The difference in point of view is well illustrated by a little incident, now mentioned in a footnote. The W-curves are defined as those having tangents cutting the triangle of reference in three points, which, with the point of contact, make a constant cross ratio. Klein was disturbed by the indeterminate form which arises when the curve has singular points at the vertices, and proposed that the paradox be investigated. (This has been done since in connection with the integrals of differential equations of the first order, by Poincaré, Vessiot, and others.) But Lie was so filled with the general theory that such things did not interest him. His reply was: "Die Kurve weiss selbst am besten, wie sie sich in singulären Punkten zu verhalten hat."

We now come to the Programm itself. It was prepared in the midst of the other early papers already mentioned, and while the author was still saturated with his work with Plücker, with Lie, with Stolz, and he was in daily contact with Clebsch while the actual paper was being written. To those who have worked a long time in algebraic geometry, most of the ideas there developed now seem almost self-evident. The idea of the group, as expressed in terms of birational transformations, the principal subgroup (Hauptgruppe), under which a surface remains invariant, the necessary restrictions on the families of curves on the surface, the change of space element, the geometric facts concerning invariants, are now the ordinary tools of the trade. But when we recall that this was written fifty years ago, that its author was twenty-three years old, that Jordan's treatise on substitution groups had hardly been out a year, that Cayley's papers on Cremona transformations, Noether's memoir on mapping and the Brill-Noether paper on algebraic curves were not then in existence, that Lie's line-sphere transformation appeared the same summer, then we realize that it was a veritable programme, which has been closely followed during the last half century. The contributions of the Italian geometers furnish a sufficiently eloquent tribute to the comprehensiveness of the plan there outlined. If we now re-read the preface of Volume III of Lie's *Theorie der Transformationengruppen*, especially page xvii, we are supplied with a great deal of food for serious reflection.

Klein remained at Erlangen five semesters, then went to Munich Easter, 1875, thence to Leipzig, October, 1880, and finally to Göttingen in 1886, at which time through his efforts Lie was made his successor at Leipzig. The first few years at Göttingen were devoted to systematizing and completing the earlier contributions, and occasionally to take a new glance at mechanics, which as a boy had interested him more than mathematics. Besides giving a course on mechanics he prepared the treatise on

the theory of the top with Sommerfeld, the first volume of which appeared in 1897. In 1901 Sir Robert Ball's *Theory of Screws* was published. The Encyklopädie was undertaken in 1894, and Klein was appointed editor of the volume on mechanics. These facts account for the next paper, a criticism of Sir Robert's book. It emphasizes what Klein had always maintained, namely, the necessity of accounting for exceptional and singular cases, to which the general theory ceases to apply. By reading the Programm and the criticism together, one sees the close connection between them. To be consistent with his own scheme, the author should have everywhere omitted the differentials of the second order, but had this been done it would lead to the exclusion of what both the author of the book and of the criticism designate as the most important part of the theory, in those cases in which the first differentials are identically zero.

The remainder of the present volume consists of much later papers; they are concerned with the connection between the theory of relativity and the projective theory, as developed in the Programm, particularly the fundamental significance of Cayley's theory of measurement. The first paper formed the closing chapter of a course on analytic projective geometry given in 1909–1910, and makes an elementary approach to the Lorentz group, showing it to be included in the affinity group in space of four dimensions. The same also contains a clear summary of the argument for the various kinds of non-euclidean geometries of fewer dimensions, and how they are related by a continuous change of the space parameter (constant of curvature). After the elementary part, entirely algebraic, there follows a short discussion of the Maxwell electro-dynamic equations, in which it is shown that they are invariant when the space coordinates are changed linearly, or the time is translated, but not under such transformations as contain both space and time coordinates in the same equation.

The next paper is a letter to Hilbert, after the appearance of Hilbert's first note on the foundations of physics. It contains an alternate derivation of Hilbert's results by means of an appropriate transformation in the sense of the Programm. Then in reply Hilbert supplies the details of the analysis, and Klein shows, by means of a system of partial differential equations that even the general theory of relativity can be treated from this same standpoint, as well as the earlier special theory. Only those consequences of the differential equations have a physical sense that are invariant under the operations of the group—a property that is inherent in the Programm for any geometry. The last two papers of the volume apply the same transformations directly to the Einstein equations in the new theory of gravitation, leading to many material simplifications. In particular, the meaning of the symbols in the multiple integration is treated at length, bringing them into clear relation with the Grassmann symbolism. The tubular or cylinder-formed universe of Einstein is contrasted with that of constant curvature of de Sitter. The general transformation theory applies directly to the latter, whereas artificial modifications must be made before it applies to the case of Einstein.

The book is printed in large clear type on good paper, and is almost completely free from typographical errors.

VIRGIL SNYDER.

SHORTER NOTICES

Lehrbuch der darstellenden Geometrie. By Dr. Georg Scheffers. Vol. I, ix + 423 pages, 404 figures. Vol. II, xiii + 439 pages, 396 figures. Berlin, Julius Springer, 1919, 1920.

This treatise on descriptive geometry contains in the main the substance of the lectures given by Professor Scheffers at the Technical School of Berlin. It presupposes nothing but "school-mathematics" on the part of the student, which, in America, is approximately equivalent to mathematical training in college algebra, trigonometry, and analytic geometry.

The courses in descriptive geometry in the universities and polytechnic schools of continental Europe are usually distinguished by high scientific standards and are given by men who are either mathematicians, like Scheffers, or men who have at least considerable mathematical knowledge beyond the calculus. The type of courses in descriptive geometry offered in most of the American engineering schools, serving a purely utilitarian purpose, may be found in the European trade-schools and secondary technical schools of various kinds, where, naturally, no effort is made to treat the subject as a science, to place connecting links with other branches of mathematics, or to pay any attention to historical developments.

From the standpoint of the scientific critic Scheffers' *Lehrbuch* is a masterly exposition of descriptive geometry. Almost on every page one may notice the enormous advantage in the treatment of the subject by the superior insight which a deeper mathematical knowledge affords. Nevertheless, Scheffers, even from the practical standpoint of the engineer, never grows pedantic, or looses himself in some individual pet schemes.

The concepts and propositions of projective geometry are developed as far as they are of unquestionable value in the treatment and rational solution of certain characteristic problems. Any one, like Scheffers, who is familiar with modern progressive views will, of course, contend that it is impossible to write a modern scientific descriptive geometry without the knowledge of projective geometry, even when taken in the larger sense of the word. The frequent lack of such knowledge on the part of teachers and writers accounts for many obsolete methods of treatment and the omission of some of the most beautiful applications of projective geometry to descriptive geometry.

The stimulating influence by Scheffers' competent selection and presentation of the subject-matter upon the reader is increased by extensive historical references and comments on geometric topics allied to descriptive and projective geometry. Throughout the treatise the author shows that he is well informed on the essential and important phases of the modern development of constructive geometric methods.

For didactic reasons, explained in the preface, the first volume starts with orthographic projection upon a single plane. The claim of modernity which the reviewer makes for Scheffers' book may, outwardly at least, appear from the fact that at the outset we find a statement about the principle of duality and its bearing upon projective processes. Further on we

find applications to the construction of roof-shaped surfaces, of sun-dials, of regular polyhedrons and crystal-forms, of ellipses and of moment-ellipses in graphic statics. The second chapter, which deals with parallel-projection upon a single plane, includes sections on cavalier-perspective, shadows under parallel light-rays, affinity and its applications, on the foundations of general axonometry (including, of course, Pohlke's famous theorem), on the theory of involution and its applications to the polar properties of the circle and the ellipse. In the third chapter we find the customary orthographic projection upon two or more planes of projection, including a valuable section on conics. Whenever possible, full use is made of the advantages which the application of projective properties of conics and the principle of affinity afford in the graphic representation of geometric forms.

The second volume contains two chapters on perspective and various applications. Chapter IV, on central projection, treats of fundamental concepts, so-called restricted perspective, construction of shadows in perspective, invariance of cross ratio, involutory perspective, applications of perspective, perspective of circle and sphere, properties of conic sections and their applications, including Pascal's and Brianchon's theorems, and so-called free perspective.

Various applications and supplementary topics, such as plane curves, surface-ornaments, topographical surfaces, surfaces of revolution, helical and cycloidal curves and surfaces, ruled surfaces, interpenetrations and shadows, and finally a brief account of relief-perspective, form the contents of the concluding fifth chapter.

The level upon which Scheffers proceeds may be judged from the fact that even a discussion of Peano's surface is included, to show the student the danger of hasty generalizations. Peano's original surface has the form $z = (y^2 - 2px)(y^2 - 2qx)$ in which p and q are positive real integers. For the sake of convenient constructive treatment, Scheffers discusses the projectively equivalent surface

$$z = -\frac{1}{10}(x^2 - 5y)(x^2 - y).$$

Every plane through the z-axis cuts the surface in a quartic which has a maximum at the origin, so that one might expect a maximum for the surface at that point. Still it is possible to trace curves on the surface passing through O having a minimum at O.

The whole treatise is carefully written and is typographically faultless. It may be heartily recommended to teachers as well as to students of descriptive geometry.

ARNOLD EMCH.

Girolamo Saccheri's Euclides Vindicatus, edited and translated by George Bruce Halsted. Chicago, The Open Court Publishing Company, 1920. 30 + 246 pp.

The original title of Saccheri's now famous work is: Euclides ab omni naevo vindicatus, . . ., which appeared in Milan in 1733, and which Halsted translates as "Euclid freed of every fleck." In English *fleck* sounds rather Teutonic, and the reviewer suspects that *flaw*, or *blemish* would sound better to the American ear. It is highly commendable that

the translator has arranged the Latin and English texts in parallel pages.

If Saccheri had drawn the last consequences of his keen logic, and if he had not unfortunately dragged in the then hazy notions of infinity and continuity, he would have established the first flawless non-euclidean geometry. On account of those concepts, foreign to euclidean geometry,* the latter portions of the book, according to Halsted, are comparatively unimportant. No less a mathematician than Segre says of this book: "Nevertheless the first seventy pages (apart from a few isolated phrases), up to Proposition 32 inclusive, constitute an ensemble of logic and of geometric acumen which may be called *perfect.*"

Considering the circumstance that the translation itself does add nothing new to well known facts, it must be said that the most valuable part of the book is the introduction written by Halsted, wherein he gives us interesting and valuable information on Saccheri's *Logica demonstrativa*. This remarkable work embodies a system of which the *Euclides Vindicatus*, which appeared later, may merely be called an application, and could obviously only originate in a mind of extraordinary ability. The first edition was published in Turin in 1697, the second in Turin in 1701, the third in Cologne in 1735. In an analysis of this logic Heath says: "Mill's account of the true distinction between real and nominal definitions was fully anticipated by Saccheri." According to Halsted, "In his *Logica demonstrativa* Saccheri lays down the clear distinction between what he calls *definitiones quid nominis* or *nominales*, and *definitiones quid rei* or *reales*, namely, that the former are only intended to explain the meaning that is to be attached to a given term, whereas the latter, besides declaring the meaning of a word, affirm at the same time the existence of the thing defined or, in geometry, the possibility of constructing it. The *definitio quid nominis* becomes a *definitio quid rei*" by means of a postulate, or when we come to the question whether the thing exists and it is answered affirmatively. "*Definitiones quid nominis* are in themselves quite arbitrary, and neither require nor are capable of proof: they are merely provisional, and are only intended to be turned as quickly as possible into *definitiones quid rei*," by means of certain postulates of existence or constructions and by means of demonstrations. Vailati, thus far the only protagonist of the *Logica demonstrativa*, says that the anticipation of Mill's distinction gives Saccheri the right to an eminent place in the history of modern logic.

On the whole, Halsted's edition of Saccheri's work is very well done and is a very praiseworthy scientific undertaking. Equally praiseworthy would be an English edition of the *Logica demonstrativa*, and we would suggest to Dr. Halsted to try to secure, for that purpose, a copy of the Cologne edition, or, for that matter, of any of the previous editions, and thus render a great service to the history of science.

ARNOLD EMCH.

* If, according to Weyl, we define as an *euclidean number* a number which is obtained from 1 by addition, subtraction, multiplication, and division, to which is adjoined the root of a positive number, then, within this system of definite content of euclidean numbers, all constructions of euclidean geometry may be carried out. It is not necessary to pour a continuous "space-broth" over it.

Géométrie Synthétique des Unicursales de Troisième Classe et de Quatrième Ordre. By E. Bally. Paris, Gauthier-Villars, 1920. 98 pp.

As its title indicates, this book develops the properties of unicursal curves of the third class and the fourth order synthetically. These curves are homographically equivalent to the tri-cuspidal hypocycloid or to the cardioid. The discussion is therefore limited to these cycloids. The ordinary cycloid is included as a limiting case.

If E and I are two concentric circles of radii R_e and R_i, respectively, $(R_e > R_i)$, there are two families of circles tangent to both whose diameters are $R_e + R_i$ and $R_e - R_i$. Consider two circles, one from each family, meeting in a point A and touching E at E_1 and E_2. If these circles roll upon E so that E_1 and E_2 move in opposite directions with velocities proportional to the radii of the rolling circles, A remains fixed in position upon each circle and describes a hypocycloid. If the circles roll upon I so that their points of contact move in the same direction with velocities proportional to the radii, A will describe an epicycloid.

The properties of the tri-cuspidal hypocycloid and of the cardioid are developed in Chapter I, starting from the foregoing mode of generation.

Chapter II deals with the tangential properties of these curves as correlative to point properties of nodal cubics.

The two following chapters deal with properties deducible from the fact that a tri-cuspidal quartic is the transform, by quadric transformation, of a conic inscribed in the triangle of its cusps.

Chapter V deals with the nodal cubic as a projection of a twisted cubic, and the tri-cuspidal quartic as a section of the developable surface formed by the tangents to the twisted cubic.

The book is interesting in that it develops the properties of these special curves in a simple way and by elementary means, but its value would be greatly enhanced to the interested reader had full references to original sources been included. For example, no reference is made to the papers of Steiner and Cremona (CRELLE, vols. 53 and 64, respectively) upon these curves. Perhaps such references will be given in the larger work on geometry which is promised in the preface, but which has not yet appeared.

L. WAYLAND DOWLING.

Éléments de Géométrie, I et II, by Alexis-Claude Clairaut. (Les Maîtres de la Pensée Scientifique.) Paris, Gauthier-Villars, 1920. 14 + 96 + 104 pp.

This is an opportune time for the re-publication of Clairaut's *Éléments de Géométrie* that was first brought out at Paris in 1741 and again in 1753. The present-day tendencies in the teaching of beginners' geometry are in some way those set forth by Clairaut 180 years ago. Many teachers will be glad to examine this eighteenth century rival of Euclid's Elements, a text written by one of the most distinguished French mathematicians of that century. At first Clairaut brings into play chiefly the intuitive powers. He secures the interest of beginners by frequent reference to exercises in surveying. The reprint appears in two small volumes.

FLORIAN CAJORI.

NOTES

The twenty-ninth summer meeting of the American Mathematical Society will be held at the University of Rochester, Rochester, New York, on Thursday and Friday, September 7 and 8. The regular sessions will be held in the Eastman Building. Kendrick Hall will be open for the accommodation of a limited number of men, and there are several hotels conveniently located. An excursion is being planned, and the usual joint dinner will be arranged with the Mathematical Association of America, which will hold its sessions immediately before those of the Society. Opportunity will be offered to those who desire to inspect the Research Laboratory of the Eastman Kodak Company, and the works of the Bausch and Lomb Optical Company.

Rochester is on the main line of the New York Central Railroad, and may be reached by several other roads.

Titles and abstracts of papers intended for presentation at the meeting should be in the hands of the secretary by August 10 in order to appear on the printed program. Papers received after that date can be presented only as time permits. This announcement replaces the preliminary notice issued in the past. The attention of members is invited to the fact that no other preliminary notice will be issued.

The editors of the AMERICAN MATHEMATICAL MONTHLY request us to announce that current issues of that journal have been long delayed not only on account of the general printing situation, which has affected all mathematical journals in America, but also still further on account of difficulties incident to transfer of editorial responsibility. These obstacles seem now to be overcome, and there is hope for gradual return to normal conditions.

A number of periodicals devoted wholly or in part to mathematics have recently been founded in various European countries. Among them are the following: ABHANDLUNGEN AUS DEM MATHEMATISCHEN SEMINAR DER HAMBURGISCHEN UNIVERSITÄT, edited by Professors W. Blaschke, E. Hecke, and J. Radon. CHRISTIAAN HUYGENS, INTERNATIONAAL MATHEMATISCH TIJDSCHRIFT, edited by Dr. F. Schuh and published by Noordhoff, at Groningen, Holland. REVISTA MATEMATICA DIN TIMIŞOARA, published under the auspices of the polytechnic school at Timisoara, Roumania, and devoted mainly to elementary mathematics and problems. SPISY VYDÁVANÉ PŘÍRODOVĚDECKOU FAKULTOU MASARYKOVY UNIVERSITY (memoirs published by the faculty of sciences of the Masaryk University), Brno (Brünn), Czecho-Slavia. WARSZAWSKIE TOWARZYSTWO POLITECHNICHE, SPRAWOZDANIA I PRACE, published by the Warsaw Polytechnic Society, and containing reports of meetings of that Society and memoirs on mathematical physics.

At the meeting of the American Association for the Advancement of Science at Toronto (December 27–31, 1921), Professor G. A. Miller was elected vice-president of Section A (mathematics) and chairman of that section. The University of Toronto conferred its honorary doctorate of

science on Professor E. H. Moore, president of the Association and an ex-president of the American Mathematical Society, on the occasion of this meeting.

The American Astronomical Union has appointed a committee on relativity consisting of Dr. Ludwik Silberstein (chairman) and Professors Edward Kasner, A. A. Michelson, and D. C. Wilson.

The British Association for the Advancement of Science will meet at Hull, September 6–13, 1922, under the presidency of Professor C. S. Sherrington. Professor G. H. Hardy is president of Section A (mathematics and physics) and Professor A. O. Rankine recorder of that section.

The London Mathematical Society has recently elected the following officers: H. W. Richmond, president; J. E. Campbell, A. L. Dickson and W. H. Young, vice-presidents; A. E. Western, treasurer; G. H. Hardy and G. N. Watson, secretaries.

The Belgian Mathematical Society (Cercle mathématique de Belgique) was founded at Brussels, October 20, 1921. T. De Donder is president.

In June, 1921, another mathematical society, Les Amis des Nombres, was founded, also at Brussels. Its aim is "to unite those who are primarily interested in numbers, both professional mathematicians and amateurs." The secretary of this society is A. Gérardin, and its official organ is the SPHINX-ŒDIPE, of which he has been editor for many years.

The Optical Society of America has arranged to bring out an English translation of Helmholtz's *Handbuch der physiologischen Optik* as a memorial of the hundredth anniversary of the author. Contributions are solicited to defray the expense of the project, and should be sent to Adolph Lomb, Treasurer of the Optical Society of America, care of Bausch and Lomb Optical Company, Rochester, N. Y.

In connection with the celebration of the twenty-fifth anniversary of the Chicago section of the American Mathematical Society, a fund has been collected, which is to be offered for trusteeship to the Society under the name of the Eliakim Hastings Moore Fund. It is proposed that the income be used for the promotion of mathematical research by the publication of important books and memoirs, and ultimately also by the award of prizes for important contributions to mathematics.

The class of sciences of the Royal Academy of Belgium has awarded prizes to P. Montel for his memoir *Sur les familles quasinormales de fonctions holomorphes* and to L. Godeaux for his memoir *Sur les transformations rationnelles de Jonquières de l'espace*. The following subjects are announced for prizes to be awarded in 1923: *An important contribution to infinitesimal geometry*, and *A contribution to the problem of bodies in the Einstein theory*.

The Paris Academy of Sciences announces the award of the following prizes in mathematics: the Francœur prize to René Baire, for his work in the general theory of functions; the Petit d'Ormoy prize to the late Georges Humbert, for his mathematical work; the Saintour prize to Pierre Boutroux, for his work in the theory of differential equations and in the history of

sciences. The Bordin prize was not awarded and the subject (in analysis situs) was withdrawn.

The Italian Society of Sciences (the XL) has awarded its prize in mathematics for 1921 to Professor O. Tedone.

The Royal Society of London has awarded its Copley medal to Sir Joseph Larmor for his work in mathematical physics, and its Hughes medal to Professor Niels Bohr, of the University of Copenhagen, for his work in theoretical physics, especially on the structure of the atom.

The gold medal of the Royal Astronomical Society has been awarded to Dr. J. H. Jeans for his researches in stellar cosmogony.

A prize for an essay in applied mathematics, the Alan Bodey prize, has been founded at Gonville and Caius College, Cambridge University. The annual value of this prize is £ 10.

The Paris Academy of Sciences has elected Professor J. Andrade a correspondent in the section of mechanics, Professor M. Brillouin a member in the section of general physics, and Professor M. d'Ocagne a member in the section of "académiciens libres."

The Accademia dei Lincei, Rome, announces the election of Professor R. Marcolongo as member in the section of pure and applied mathematics, and of Professor G. Armellini as corresponding member and Professors A. Einstein and C. de la Vallée Poussin as foreign associates in that section.

The Italian Society of Sciences (the XL) has elected Professors E. Almansi and G. Ricci to membership.

Professor Niels Bohr has been elected a member of the Royal Institution, London.

The following advanced courses are announced for the summer session of 1922:

UNIVERSITY OF CHICAGO: First term, June 19–July 26; second term, July 27–September 1. By Professor E. H. Moore: Fundamental number systems of analysis, first term; Limits and series, first term. By Professor H. E. Slaught: Elliptic integrals; Differential equations. By Professor G. A. Bliss: Functions of a complex variable; Calculus of variations. By Professor E. J. Wilczynski: Metric differential geometry. By Professor J. W. A. Young: Theory of equations. By Professor F. R. Moulton: Celestial mechanics. By Professor A. C. Lunn: Statistics and probability; vector analysis. By Professor R. L. Moore: Foundations of analysis situs. By Mr. H. S. Vandiver: Theory of numbers.

Professor J. Nielsen, of the Breslau Technical School, has been appointed to a professorship at the Academy of Agriculture at Copenhagen.

Dr. R. Wavre has been admitted as privat-docent at the University of Geneva.

Dr. K. Eisenmann has been appointed professor of technical mechanics at the Braunschweig Technical School.

In the faculty of sciences of the University of Paris, Professor J. Drach has been transferred from the chair of general mathematics to that of the

application of analysis to geometry, and Professor E. Cotton from the chair of theoretical physics to that of general physics (as successor to the late Professor G. Lippmann).

The following appointments to professorships in other French universities are announced: Dr. Darmois (higher analysis) at the University of Nancy; Dr. Deltheil (general mathematics) at the University of Toulouse; Dr. G. Giraud (differential and integral calculus) at the University of Clermont; Dr. P. Humbert (mathematics) at the University of Montpellier; Dr. J. Pérès (rational mechanics) at the University of Montpellier; Dr. E. Turrière (rational mechanics) at the University of Aix-Marseilles.

At the University of Rome, Professor G. Bagnera, of the University of Palermo, has been appointed professor of infinitesimal analysis, Professor F. Severi, of the University of Padua, professor of algebraic analysis, and Professor F. Enriques, of the University of Bologna, professor of mathematical methodology; and Dr. Zondadari has been admitted as privatdocent in descriptive geometry.

Professor U. Cisotti, of the University of Pavia, has been appointed professor of mathematical physics at the Milan Technical Institute.

Dr. J. H. Jeans, secretary of the Royal Society of London, has been appointed Halley lecturer at Oxford University for 1922.

Mr. W. E. H. Berwick, university lecturer at Cambridge, has been appointed to a fellowship at Clare College.

Mr. H. W. Turnbull has been appointed regius professor of mathematics at the University of St. Andrews, as successor to Sir P. S. Leng, who has resigned.

Dr. J. W. Nicholson, formerly of the University of London, has been appointed to a war memorial fellowship as tutor in mathematics and physics at Balliol College, Oxford.

Dr. L. L. Dines has been promoted to a senior professorship of mathematics at the University of Saskatchewan, and granted leave of absence for the second half of the present academic year.

Mr. F. W. Winters has been appointed assistant professor of mathematics at Dalhousie College, Halifax, Nova Scotia.

Professor F. S. Nowlan, of Bowdoin College, has been appointed lecturer in mathematics at the University of Manitoba.

Dr. J. J. Nassau has been appointed assistant professor of mathematics and astronomy at the Case School of Applied Science.

Miss Elsie M. Plapp, of the University of Wisconsin, has been appointed adjunct professor of mathematics at Hollins College.

Mrs. H. B. Newson, formerly assistant professor of mathematics at Washburn College, has been appointed professor of mathematics at Eureka College.

Associate Professor Emma L. Konantz has returned to Ohio Wesleyan University after two years leave of absence spent in teaching at Peking University.

At the University of Denver, Mr. B. F. Dostal, of Oberlin College, has been appointed assistant professor of mathematics.

Mr. Ralph Beatley, of Horace Mann School and Teachers College, Columbia University, has been appointed assistant professor of education at Harvard University; his work will lie in the field of the teaching of mathematics, and he will also give one course in the department of mathematics.

Dr. F. C. Touton has been appointed lecturer in secondary education at the University of California.

Mrs. Ethelwynn R. Beckwith, formerly assistant professor of mathematics at the College for Women, Western Reserve University, has been appointed acting assistant professor of mathematics at Vassar College.

Mr. J. C. Funk has been appointed head of the department of mathematics at the Santa Maria high school and junior college.

At the U. S. Naval Academy, Mr. G. F. Abrich, Mr. R. P. Johnson, Mr. R. C. Lamb, Mr. E. S. Mayer, and Mr. J. B. Scarborough have been promoted from instructorships to assistant professorships of mathematics.

Dr. Philip Franklin has been appointed Benjamin Peirce Instructor at Harvard University for the year 1922–23.

Recent appointments as instructor in mathematics in American colleges and universities are as follows: Brown University, Mr. H. K. Cummings; Syracuse University, Mr. I. S. Carroll; University of Colorado, Dr. Claribel Kendall (returned from leave of absence); University of Michigan, Mr. J. P. Ballantine, and Mr. W. M. Coates.

The death of Professor Charles Cailler of the University of Geneva is reported.

Professor Eduard Gubler, of Zürich, died November 6, 1921, at the age of seventy-six years.

Professor H. J. Cotterill, of the Royal Naval College, Greenwich, died January 8, 1922, at the age of eighty-six years.

Professor W. Foord-Kelcey, of the Royal Military Academy, Woolwich, died January 3, 1922, at the age of sixty-seven years.

Professor C. J. Lambert, of the Royal Naval College, Greenwich, died November 11, 1921, at the age of seventy-seven years.

Rev. J. B. Lock, formerly lecturer on mathematics and tutor at Caius College, Cambridge, died September 8, 1921, at the age of seventy-two years.

Mr. George Wentworth, co-author with Professor D. E. Smith of the Wentworth and Smith series of mathematical text-books, died August 26, 1921, at the age of fifty-three years. Mr. Wentworth had been a member of the American Mathematical Society since 1910.

The death of Professor W. W. Beman, previously announced in this BULLETIN (vol. 28, p. 82) occurred on January 18, 1922, not 1921 as there stated.

NEW PUBLICATIONS

I. HIGHER MATHEMATICS

APPELL (P.). Eléments de la théorie des vecteurs et de la géométrie analytique. Paris, Payot, 1921. 16mo. 147 pp. Fr. 4.00

ARWIN (A.). Ueber Kongruenzen von den fünften und höheren Graden nach einem Primzahl-modulus. Stockholm, 1918. 8vo. 46 pp.

BACHMANN (P.). Grundlehren der neueren Zahlentheorie. 2te, verbesserte Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 8vo. 16 + 252 pp.

BIANCHI (L.). Lezioni di geometria analitica. Pisa, Spoerri, 1920. 8vo. 604 pp. L. 40.00

BIEBERBACH (L.). Lehrbuch der Funktionentheorie. Band 1: Elemente der Funktionenlehre. Leipzig, Teubner, 1921. 8vo. 6 + 314 pp.

—— See ENCYKLOPÄDIE.

BONOLA (R.). Die nichteuklidische Geometrie. Historisch-kritische Darstellung ihrer Entwicklung. Deutsche Ausgabung von H. Liebmann. 3te Auflage. Leipzig, 1921.

BURKHARDT (H.). Funktionentheoretische Vorlesungen. Band 1, Heft 1: Algebraische Analysis. 3te Auflage. Band 1, Heft 2: Einführung in die Theorie der analytischen Funktionen einer komplexen Veränderlichen. 5te Auflage. Band 2: Elliptische Funktionen. 3te Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1920–1921. 8vo.

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